Example: Is the zero matrix in reduced row-echelon form? Yes.

Example: How many solutions has each system below?

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$x + 2y = 0 \rightsquigarrow x = -2y$

No solution. Infinitely many. One solution.

A system of equations is called <u>consistent</u> if it has at least one solution. It is called <u>inconsistent</u> if it has no solutions.

Theorem: A linear system is inconsistent if and only if its reduced row-echelon form has a row of the form $[0 \cdots 0 \mid 1]$. If a linear system is consistent then:

(i) it has infinitely many solutions if there is at least one free variable.

(ii) it has exactly one solution if all the variables are leading.

The <u>rank</u> of a matrix is the number of leading 1's in its rref.

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ The rank is 2.

Theorem: Consider a system with n equations and m variables. Then: $(n \times m)$

is coefficient matrix

(i) We have $\text{rank}(A) \le n$ and $\text{rank}(A) \le m$.

(ii) If $\text{rank}(A) = n$, then the system is consistent.

(iii) If $\text{rank}(A) = m$, then the system has at most one solution.

(iv) If $\text{rank}(A) < m$, then the system has zero or infinitely many solutions.

Why ?

Example:

1. Suppose we have a system with fewer equations than variables.

   How many solutions may it have ? Zero or infinitely many.

2. Suppose that a system has n equations and n variables.

   When do we have exactly one solution ? Rank n.

Addition: $C = A + B \qquad c_{ij} = a_{ij} + b_{ij}$

Scalar multiplication: $C = kA \qquad c_{ij} = k\, a_{ij}$

Dot product: $\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i \qquad [x_1 \cdots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

Multiplication of matrix and vector:

$$A\vec{x} = \begin{array}{r} \\ \\ \mathbb{R}^m \rightarrow \end{array} \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \underset{\substack{\uparrow \\ \mathbb{R}^m}}{\vec{x}} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$$

$\mathbb{R}^n$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} [1\ 2]\cdot\begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ [3\ 4]\cdot\begin{bmatrix} 5 \\ 6 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + 2\cdot 6 \\ 3\cdot 5 + 4\cdot 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = 5\cdot\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6\cdot\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 12 \\ 24 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

<u>Algebraic rules:</u> $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ and $A(k\vec{x}) = kA\vec{x}$

A vector $\vec{v}$ is a <u>linear combination</u> of $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ if there are scalars $a_1, \dots, a_m$ such that $\vec{v} = a_1\vec{v}_1 + \cdots + a_m\vec{v}_m$.

Given a linear system with augmented matrix $[A \mid \vec{b}]$, we can write it as an equality of matrices $A\vec{x} = \vec{b}$ where $\vec{x}$ is the vector of variables.

<u>Example:</u>

$$\begin{aligned} 2x + 8y + 4z &= 2 \\ 2x + 5y + z &= 5 \\ 4x + 10y - z &= 1 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \rightsquigarrow \begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

<u>Example:</u>

1. There is a 3x3 matrix of rank 4. F.

2. There is a system of 3 eqs, 3 unk, with 3 sols. F.

3. If A is a 3×4 matrix of rank 3, then $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ must have infinitely many solutions. True!

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R_3 - 7 \cdot R_1]{R_2 - 4 \cdot R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{1}{-3}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix}$$

$$\xrightarrow{R_3 + 6 \cdot R_2 \quad R_1 - 2 \cdot R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$