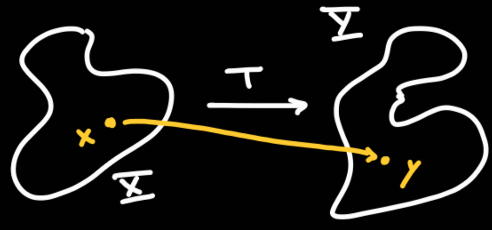
A <u>function</u> T from $\mathbb{X}$ to $\mathbb{Y}$ is an assignment of an unique element y of $\mathbb{Y}$ to an element x in $\mathbb{X}$. $T(x) = y$

upper case



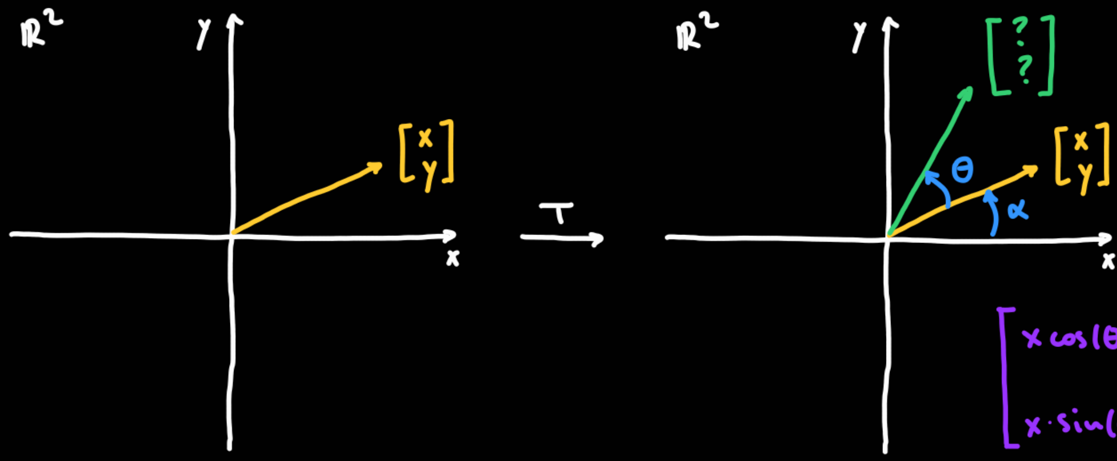
A <u>linear transformation</u> is a function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ such that there exists a matrix A that is $n \times m$ with $T(\vec{x}) = A\vec{x}$.

matrix associated to the linear transformation T

$$\underset{2\times 3}{\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}} \underbrace{\begin{bmatrix} * \\ * \\ * \end{bmatrix}}_{\mathbb{R}^3} = \underbrace{\begin{bmatrix} * \\ * \end{bmatrix}}_{\mathbb{R}^2}$$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}} \underbrace{\begin{bmatrix} * \\ * \end{bmatrix}}_{\mathbb{R}^2} = \underbrace{\begin{bmatrix} * \\ * \\ * \end{bmatrix}}_{\mathbb{R}^3}$$

<u>Example:</u> Consider the function from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by a rotation of angle θ.



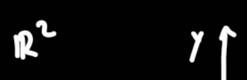
$$\begin{bmatrix} y \\ x \end{bmatrix}$$

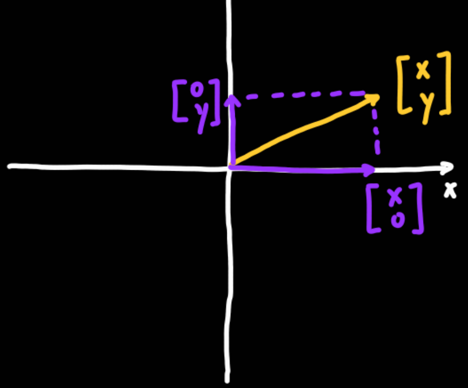
$$\begin{bmatrix} \sqrt{x^2+y^2}\cos(\theta) \\ \sqrt{x^2+y^2}\sin(\theta) \end{bmatrix}$$

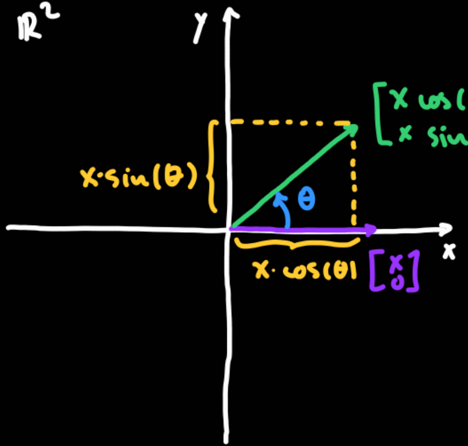
$$\begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\cdot\sin(\theta) + y\cos(\theta) \end{bmatrix}$$

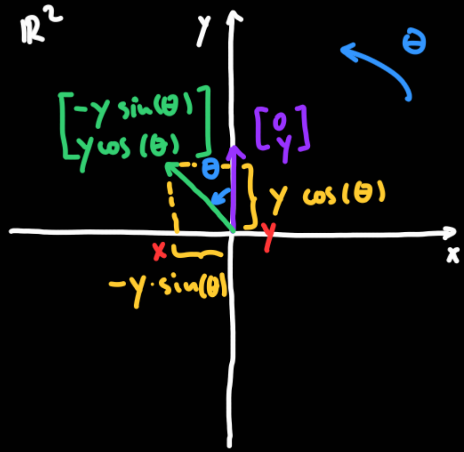
This rotation is a linear transformation!

$$T(\vec{x}) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\text{matrix associated to } T} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}$$







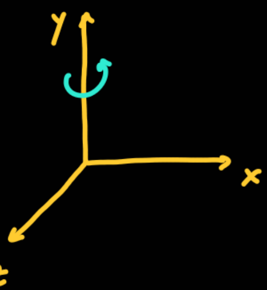


$$\begin{bmatrix} x\cos(\theta) \\ x\sin(\theta)\end{bmatrix} + \begin{bmatrix} -y\sin(\theta) \\ y\cos(\theta)\end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta)\end{bmatrix}.$$

Theorem: Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. The columns of the matrix associated to T are:

$$\begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_m) \\ | & & | \end{bmatrix}, \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

i-th component



T is from $\underbrace{\mathbb{R}^3}_{\text{columns}}$ to $\underbrace{\mathbb{R}^3}_{\text{rows}}$

Theorem: A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a linear transformation if.o.if:

(i) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ and

(ii) $T(\lambda \vec{v}) = \lambda T(\vec{v})$ $\qquad \lambda$ scalar in $\mathbb{R}$

lambda

<u>Example</u>: Of non-linearity: $T(x) = x^2$.

$\lambda = 2$, $x = 3$ $\quad T(2 \cdot 3) = 36 \neq 2 \cdot T(3) = 2 \cdot 9 = 18$