Recall: We could do consecutive linear transformations: we can scale the plane, and

we want to formalize this

then rotate it. This is also a linear transformation!

How do we deal with "composition/concatenation" of linear transformations?

Let T be a linear transformation given by $\begin{bmatrix}6&7\\8&9\end{bmatrix}$, let S be a linear transformation given by $\begin{bmatrix}1&2\\3&5\end{bmatrix}$. We would like to find $\vec{z} = \underbrace{T}_{\text{2nd}}(\underbrace{S(\vec{x})}_{\text{1st}})$. We do this in two steps, first we find $\vec{y} = S(\vec{x})$, then we find $\vec{z} = T(\vec{y})$.

$\vec{y} = S(\vec{x})$ is $\begin{bmatrix}y_1\\y_2\end{bmatrix} = \underbrace{\begin{bmatrix}1&2\\3&5\end{bmatrix}}_{S}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, so
$$y_1 = x_1 + 2x_2$$
$$y_2 = 3x_1 + 5x_2$$

$\vec{z} = T(\vec{y})$ is $\begin{bmatrix}z_1\\z_2\end{bmatrix} = \underbrace{\begin{bmatrix}6&7\\8&9\end{bmatrix}}_{T}\begin{bmatrix}y_1\\y_2\end{bmatrix}$, so
$$z_1 = 6y_1 + 7y_2$$
$$z_2 = 8y_1 + 9y_2$$

substitute y_1, y_2

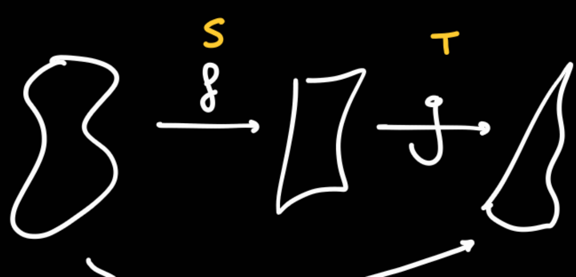
$$z_1 = 27x_1 + 47x_2$$
$$z_2 = 35x_1 + 61x_2$$

$$\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}27&47\\35&61\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

matrix equation associated to

This final system also has an equation in matrix form!

This should mean that $\vec{z} = TS(\vec{x})$ is given by $\begin{bmatrix}27&47\\35&61\end{bmatrix}$. This matrix should be the product of the matrices for T and S, i.e. $\underbrace{\begin{bmatrix}6&7\\8&9\end{bmatrix}}_{T}\underbrace{\begin{bmatrix}1&2\\3&5\end{bmatrix}}_{S}$.



$g \circ f$

TS

Matrix multiplication:

columns $\qquad\qquad$ rows

Let B be an $n \times p$ matrix and A a $q \times m$ matrix. If (and only if) $p = q$ the product BA is the matrix associated to the linear transformation $T(\vec{x}) = B(A\vec{x})$. This will be an $n \times m$ matrix.

$$\begin{array}{ccc} \vec{x} & A\vec{x} & B(A\vec{x}) \\ \mathbb{R}^m \xrightarrow{A} & \mathbb{R}^q & \\ & \mathbb{R}^p \xrightarrow{B} & \mathbb{R}^n \end{array}$$

Theorem: Let B be an $n \times p$ matrix and A a $p \times m$ matrix. Then:

(i) $$BA = B\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ B\vec{v}_1 & \cdots & B\vec{v}_m \\ | & & | \end{bmatrix}$$

(ii) $$C = BA = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix}\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & c_{ij} & \cdots \\ & \vdots & \end{bmatrix}$$

has entries $c_{ij} = \vec{w}_i \cdot \vec{v}_j = \sum_{k=1}^{p} b_{ik} a_{kj}$.

$$\vec{w}_i = [b_{i1} \cdots b_{ip}] \qquad \vec{v}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{pj} \end{bmatrix}$$

Example: Matrix multiplication is not commutative.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$$

Algebraic rules:

$m \times m$ identity matrix

(i) For any $n \times m$ matrix A, we have: $A I_m = A = I_n A$.

$n \times n$ identity matrix

(ii) Associativity: $(AB)C = A(BC)$.

(iii) Distributivity: $(A+B)C = AC + BC$ and $A'(B'+C') = A'B' + A'C'$.

(iv) Multiplication by scalars can be factored out:

$$(kA)B = k(AB) = A(kB)$$