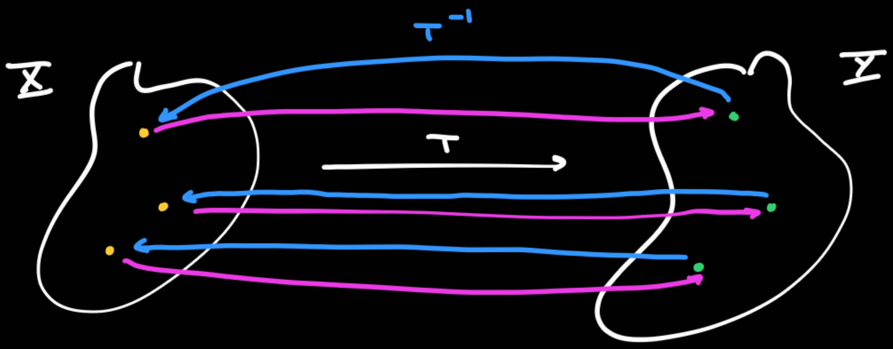
Recall: A function T from X to Y is called **invertible** if for each y in Y

there is a unique element x in X with $T(x)=y$.



inverse T^{-1} sends y to x: $T^{-1}(y)=x$ i.f.o.i.f $T(x)=y$

A function T has inverse $L=T^{-1}$ if and only if:

$T(L(y))=y$ for all y, and $L(T(x))=x$ for all x.

A square matrix A is said to be **invertible** if the associated linear transformation

$\vec{y}=T(\vec{x})=A\vec{x}$ is invertible.

T^{-1} will also be a linear transformation, its associated matrix is A^{-1}!

Theorem: Let A be an $n\times n$ matrix.

n×n identity matrix

(i) A is invertible if and only if $\text{rank}(A)=n$, i.f.o.i.f $\text{rref}(A)=I_n$.

(ii) If A is invertible then the equation $A\vec{x}=\vec{b}$ has exactly one solution

$\vec{x}=A^{-1}\vec{b}$.

Example: A $n\times n$ matrix. The equation $A\vec{x}=\vec{0}$ always has $\vec{x}=0$ as a solution.

If A is invertible then $\vec{x} = \vec{0}$ is the only solution.

If A is <u>not</u> invertible then $A\vec{x} = \vec{0}$ has infinitely many solutions.

<u>Example:</u> Is rotating by θ counterclockwise an invertible transformation? Yes, it has inverse rotating by θ clockwise.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$-\theta$ counterclockwise

<u>Theorem:</u> To find the inverse of an $n \times n$ matrix A, compute $\operatorname{rref}([A \mid I_n])$.

(i) If $\operatorname{rref}([A \mid I_n]) = [I_n \mid B]$ then A is invertible and $B = A^{-1}$.

(ii) If $\operatorname{rref}([A \mid I_n]) \neq [I_n \mid B]$ then A is <u>not</u> invertible.

<u>Example:</u> The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} = A$ is invertible because:

$$\operatorname{rref}\left(\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & 1 \end{array}\right].$$

A^{-1}

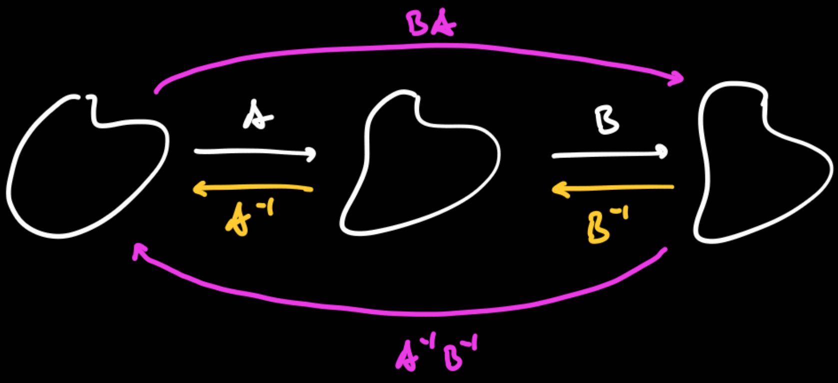
$$[A \mid I_n] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 5R_2]{R_1 - R_2}$$

<u>Theorem:</u> If A, B are invertible $n \times n$ matrices, then:

(i) $A A^{-1} = I_n$ and $A^{-1} A = I_n$.

$T(L(y)) = y$ $\qquad$ $L(T(x)) = x$

(ii) BA is invertible with inverse $(BA)^{-1} = A^{-1} B^{-1}$.



Example: Multiply the matrices we found!

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \cos^2\theta + \sin^2\theta = 1.$$

The 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has determinant $\det(A) = ad - bc$. The matrix

$A\vec{x} = \vec{y}$

$$x_1 = \frac{d \cdot y_1}{ad-bc} + \frac{-b \cdot y_2}{ad-bc}$$

$$x_2 = \frac{-c \cdot y_1}{ad-bc} + \frac{a \cdot y_2}{ad-bc}$$

A is invertible i.f.o.i.f $\det(A) \neq 0$.

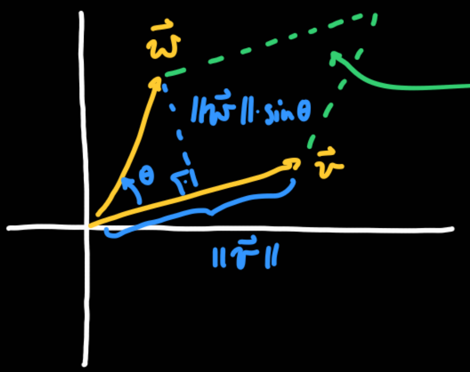
If A is invertible $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\left(A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) \right).$

Example: For which values of k is $A = \begin{bmatrix} k-1 & 2 \\ 4 & 3-k \end{bmatrix}$ invertible?

We have $\det(A) = (k-5)(k+1)$, and $\det(A) \neq 0$ iff A invertible,

we have that A is invertible as long as $k \neq 5, -1$.

$A = [\vec{v} \; \vec{w}]$



$$\text{Area} = \text{base} \cdot \text{height} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin\theta = \det(A)$$

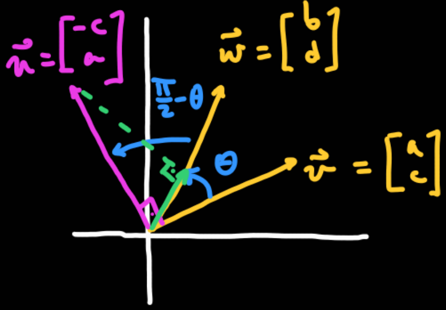
$$= v_1 \cdot w_2 - v_2 \cdot w_1$$

$$\text{Area base} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \sin\theta$$

$$\text{Volume} = \underbrace{\|\vec{v}\|}_{\vec{v}} \cdot \underbrace{\|\vec{w}\| \cdot \sin\theta}_{\substack{\text{orthogonal projection} \\ \vec{w}^{\perp}}} \cdot \underbrace{\|\vec{u}\| \cdot \sin\varphi}_{\substack{\text{orthogonal projection} \\ \vec{u}^{\perp}}} =$$

$$= \det([\vec{v}\ \vec{w}]) \cdot \|\vec{u}^{\perp}\| .$$

$\vec{u}$ is $\vec{v}$ rotated $\frac{\pi}{2}$.



$$A = [\vec{v}\ \ \vec{w}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc = \underbrace{\vec{u} \cdot \vec{w}}_{\substack{\text{projecting } \vec{u} \text{ onto } \vec{w} \text{ and scaling} \\ \text{by the length of } \vec{w}.}} = \|\vec{u}\| \cdot \cos\left(\frac{\pi}{2} - \theta\right) \cdot \|\vec{w}\| = \|\vec{v}\| \cdot \sin\theta \cdot \|\vec{w}\|$$