2. Subspaces of $\mathbb{R}^n$.

$$T: X \longrightarrow Y$$
im(T) lives here

The *image* of a linear transformation T is set of values of the function.

$$\text{im}(T) = \{ T(x) \mid x \in X \} = \{ y \in Y \mid y = T(x) \text{ for some } x \in X \}.$$

such that

curly bracket

in, "belongs to"

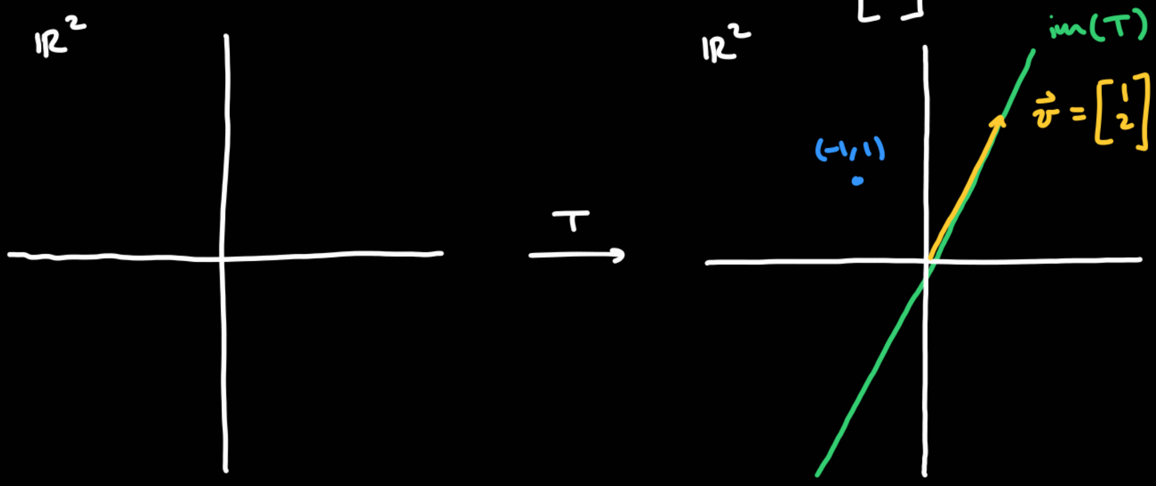
Example:

1. $\underset{\text{source}}{T: \mathbb{R}^2} \longrightarrow \underset{\text{target}}{\mathbb{R}^2}$ given by $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. To find the image, we apply T to $\vec{x}$:

$$T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$= (x_1 + 3x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So im(T) is scalar multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$: $\text{im}(T) = \left\{ k \begin{bmatrix} 1 \\ 2 \end{bmatrix} \,\middle|\, k \in \mathbb{R} \right\}$.



(-1,1) is in $\mathbb{R}^2$

(-1,1) is <u>not</u> in im(T)

A invertible. pick $\vec{y} \in \mathbb{R}^2$ (target)

then we want $\vec{x} \in \mathbb{R}$ (source) with

$A\vec{x} = \vec{y}$.

$\vec{x} = A^{-1}\vec{y}$.

$\text{im}(A) = \mathbb{R}^2$

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad$ $\text{im}(T) = \begin{cases} 1. \ \vec{0} \\ 2. \text{ one line through } \vec{0} \\ 3. \ \mathbb{R}^2 \end{cases}$

"subspaces of $\mathbb{R}^2$"

Theorem: Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then:

(i) $\vec{0} \in \operatorname{im}(T)$, $\vec{0} \in \mathbb{R}^n$.

(ii) If $\vec{v}_1, \vec{v}_2 \in \operatorname{im}(T)$ then $\vec{v}_1 + \vec{v}_2 \in \operatorname{im}(T)$.

(iii) If $\vec{v} \in \operatorname{im}(T)$ then $k\vec{v} \in \operatorname{im}(T)$ for any $k \in \mathbb{R}$.

$\mathbb{R}^n$

$\mathbb{R}^{n-1}$

$\vdots$

$\mathbb{R}^2$

$\mathbb{R}^1$

$\mathbb{R}^0 = \{\vec{0}\}$

These properties will later define "subspaces"!

Let $\vec{v}_1, \dots, \vec{v}_m$ be vectors in $\mathbb{R}^n$. The set of all their linear combinations is called their <u>span</u>:

$$\operatorname{span}(\vec{v}_1, \dots, \vec{v}_m) = \{ c_1\vec{v}_1 + \dots + c_m\vec{v}_m \mid c_1, \dots, c_m \in \mathbb{R} \}$$

The image of a linear transformation is the span of its columns.

$$T: \underset{\ker(T)}{X} \longrightarrow \underset{\operatorname{im}(T)}{Y}$$

$$\underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}}_{T}, \text{ then } \operatorname{im}(T) = \operatorname{span}(\vec{v}_1, \dots, \vec{v}_m).$$

The <u>kernel</u> of $T: \mathbb{R}^m \to \mathbb{R}^n$ is the set of values that the function sends to $\vec{0}$.

$$\ker(T) = \{ \vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0} \},$$ the solutions of the equation $T(\vec{x}) = \vec{0}$.

Example: Find the kernel of $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

We need to solve $T(\vec{x}) = \vec{0}$. This is:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 0 \end{aligned} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$x_1 = x_3, \quad x_2 = -2x_3.$

Set $x_1 = t$ the free variable, then:

$$\vec{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \text{so} \quad \ker(T) = \left\{ k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \,\middle|\, k \in \mathbb{R} \right\}.$$

Example: T invertible linear transformation. If $\vec{x} \in \ker(T)$ then $T(\vec{x}) = \vec{0}$, so
A associated matrix

$A\vec{x} = \vec{0}$. Applying A^{-1}: $\vec{0} = A^{-1}\vec{0} = A^{-1}(A\vec{x}) = \vec{x}$. So $\ker(T) = \{\vec{0}\} = 0$.

Example: A $n \times m$ matrix with $\ker(A) = \overset{=\{\vec{0}\}}{0}$. Then the system $A\vec{x} = \vec{0}$ has

because the kernel of A has one single element.

exactly one solution. Thus, there are no free variables. Thus all my variables

are leading. Thus $\operatorname{rank}(A) = m$.

A $n \times m$ $\quad n \rightsquigarrow$ equations $\quad$ $n \times n$ matrix with rank n

$\qquad\qquad\quad m \rightsquigarrow$ variables $\quad$ is invertible.

Note: The previous theorem is true if we replace image by kernel.

Theorem: A $n \times m$ matrix $\qquad$ **Notation:** $\{\vec{0}\} = 0$.

(i) $\ker(A) = 0$ iff $\operatorname{rank}(A) = m$.

(ii) If $\ker(A) = 0$ then $m \leq n$.

(iii) If $m > n$ then there are non-zero vectors in $\ker(A)$

(iv) Let $n = m$. $\quad \ker(A) = 0$ iff A is invertible.

Linear combination: $v_1, \dots, v_m$ pick some scalars $c_1, \dots, c_m$,

multiply $c_1\vec{v}_1, c_2\vec{v}_2, \dots, c_m\vec{v}_m$, then add them up:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m.$$