A subset W of $\mathbb{R}^n$ is called a <u>linear subspace</u> if:

(i) $\vec{0} \in W$

(ii) $\vec{v}_1, \vec{v}_2$ in W then $\vec{v}_1 + \vec{v}_2$ is in W.

(iii) $\vec{v} \in W$, $k \in \mathbb{R}$ then $k \cdot \vec{v} \in W$

<u>Example:</u>

1. $\mathbb{R}^2$

2. $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ n rows, m columns

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$\ker(T)$ $\quad$ $\operatorname{im}(T)$

3. $\vec{0}$

4. Let V be a plane in $\mathbb{R}^3$ given by $x + y + z = 0$.

($x + y + z = 1$ is <u>not</u> a subspace since $\vec{0} \in \mathbb{R}^3$ is <u>not</u> in it)

(a) Find a matrix A such that $\ker(A) = V$.

($V =$) $\ker(A)$ is given by solutions to the system $A\vec{x} = \vec{0}$. ($= x + y + z$)

Since V is in $\mathbb{R}^3$, we need A to input a vector in $\mathbb{R}^3$.

Since V is given by one equation, we need A to output something in $\mathbb{R}$.

rows output

columns input

A is then a 1×3 matrix.

$$0 = A\vec{x} = \begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz$$

$$0 = ax + by + cz \longleftrightarrow 0 = x + y + z \quad \text{so } a=1, b=1, c=1.$$

So $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ has kernel V.

(b) Find a matrix B such that $\operatorname{im}(B) = V$.

"The image of B is the linear combination of its columns".

$$B = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \qquad \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \underbrace{x_1\cdot\vec{v}_1 + \cdots + x_m\cdot\vec{v}_m}_{\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m).}$$

If we take two vectors in V that span all of V, and we put them in the columns of B, then $\operatorname{im}(B) = V$.

$$V: \quad x + y + z = 0$$

$$\begin{matrix} z = 0 \\ y = 1 \end{matrix} \rightsquigarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{matrix} y = 0 \\ z = 1 \end{matrix} \rightsquigarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

both vectors are in V, they are not parallel

So $B = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ has $\operatorname{im}(B) = V$.

Example:

$$A = \begin{array}{cccc} & & & \\ \end{array}\begin{bmatrix} 1 & 3 & 1 & 3 \\ 1 & 3 & 2 & 4 \\ 1 & 3 & 3 & 5 \end{bmatrix}$$

$$\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array}$$

$\vec{v}_2 = 3\cdot\vec{v}_1$

$\vec{v}_4 = 2\cdot\vec{v}_1 + \vec{v}_3$

$\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$ is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_m$

$$\text{im}(A) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \left\{ c_1 \cdot \vec{v}_1 + \underline{c_2 \cdot \vec{v}_2} + c_3 \cdot \vec{v}_3 + \underline{c_4 \cdot \vec{v}_4} \right\} =$$

$$= \left\{ c_1 \vec{v}_1 + c_2 \cdot 3 \cdot \vec{v}_1 + c_3 \vec{v}_3 + c_4 \cdot 2 \cdot \vec{v}_1 + c_4 \cdot \vec{v}_3 \right\} =$$

$$= \left\{ (c_1 + 3c_2 + 2c_4) \cdot \vec{v}_1 + (c_3 + c_4) \cdot \vec{v}_3 \right\} = \text{span}(\vec{v}_1, \vec{v}_3).$$

Let $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$. We say that $\vec{v}_i$ is <u>redundant</u> if it is a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}$. If none of them is redundant, we say that $\vec{v}_1, \dots, \vec{v}_m$ are <u>linearly independent</u>.

$$v_1 + v_3 = v_2 \longrightarrow v_1 - v_2 = -v_3 \longrightarrow -v_1 + v_2 = v_3$$

If $\vec{v}_1, \dots, \vec{v}_m$ in a subspace V, span V and are linearly independent, we call them a <u>basis</u> of V.

$$V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$$

<u>Example:</u> Consider $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$. Are these vectors linearly independent?

Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$. Compute $\underbrace{\ker(A)}_{A\vec{x} = \vec{0}}$.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x_3 = t$$

$$\vec{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \text{so} \quad \ker(A) = \text{span}\left(\underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}\right).$$

So: basis of ker(A)

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix} = 1\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\cdot\begin{bmatrix}4\\5\\6\end{bmatrix} + 1\cdot\begin{bmatrix}7\\8\\9\end{bmatrix}$$

$$\begin{bmatrix}7\\8\\9\end{bmatrix} = -\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}.$$

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right) = \operatorname{span}\underbrace{\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right)}_{\text{basis of } \operatorname{im}(A)}$$

<u>Theorem:</u> Let $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$. The following are equivalent: (one happens if and only if another happens)

(i) $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.

(ii) None of $\vec{v}_1, \dots, \vec{v}_m$ is redundant.

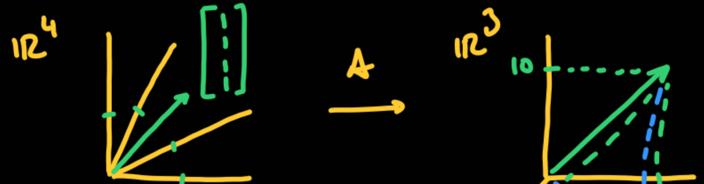
(iii) None of $\vec{v}_1, \dots, \vec{v}_m$ is a linear combination of the others.

(iv) $\ker\left(\begin{bmatrix}| & & |\\ \vec{v}_1 & \cdots & \vec{v}_m\\ | & & |\end{bmatrix}\right) = 0$

(v) $\operatorname{rank}\left(\begin{bmatrix}| & & |\\ \vec{v}_1 & \cdots & \vec{v}_m\\ | & & |\end{bmatrix}\right) = m.$

$$A = \begin{bmatrix}1&3&1&3\\1&3&2&4\\1&3&3&5\end{bmatrix} \qquad T: \underset{\text{columns}}{\mathbb{R}^4} \longrightarrow \underset{\text{rows}}{\mathbb{R}^3}$$

$$A\vec{x} \qquad \vec{x} = \begin{bmatrix}*\\*\\*\\*\end{bmatrix} \qquad A\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}8\\10\\12\end{bmatrix} \qquad \operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right).$$




8
12