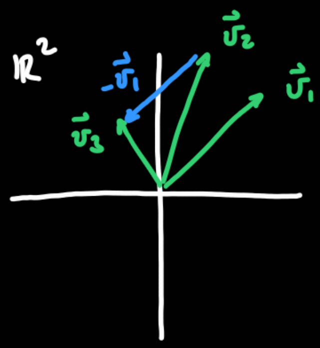
Recall: The vectors $\vec{v}_1, \dots, \vec{v}_m$ form a basis of V a subspace of $\mathbb{R}^n$ when:

(i) $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ and

(ii) $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.



$\mathbb{R}^2 = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \text{span}(\vec{v}_1, \vec{v}_2) = \text{span}(\vec{v}_1, \vec{v}_3)$

$\vec{v}_3 = \vec{v}_2 - \vec{v}_1$

Theorem: Let $\vec{v}_1, \dots, \vec{v}_m$ be vectors in V. They form a basis of V if and only if

every vector in V can be written as a linear combination:

$\vec{v} = \underline{c_1} \cdot \vec{v}_1 + \dots + \underline{c_m} \cdot \vec{v}_m$ in a unique way.

coordinates of $\vec{v}$ with respect to $\vec{v}_1, \dots, \vec{v}_m$

Theorem: Let $\vec{v}_1, \dots, \vec{v}_m$ be a basis of V. Then:

(i) Any spanning set of V has at least m vectors.

$V = \text{span}(\vec{w}_1, \dots, \vec{w}_j)$ then $j \geq m$.

(ii) Any other basis has m elements.

The number $\dim(V)$ of vectors in a basis of V is called the dimension of V.

Example: Set $\mathbb{R}^3$ the ambient space. Let V a subspace of $\mathbb{R}^3$.

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ V can have at most 3 linearly independent vectors.

Theorem: Let V be a subspace of dimension $m = \dim(V)$.

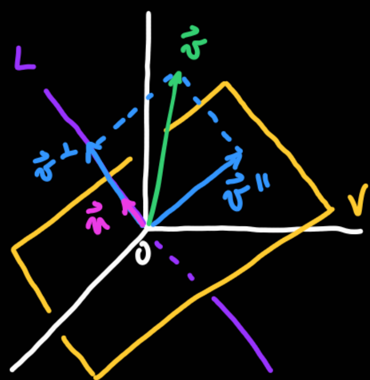
(i) There are at most m linearly independent vectors in V.

(ii) We need at least m vectors to span V.

(iii) If $V = \text{span}(\vec{w}_1, \dots, \vec{w}_m)$ then $\vec{w}_1, \dots, \vec{w}_m$ are linearly independent. (so they form a basis).

(iv) If $\vec{w}_1, \dots, \vec{w}_m$ are linearly independent in V then $V = \text{span}(\vec{w}_1, \dots, \vec{w}_m)$ (so they form a basis).

Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto V.



$\text{im}(T) = V$ so $V = \text{span}(\vec{w}_1, \vec{w}_2)$ so $\dim(V) = 2$.

two non-parallel vectors in V

$\ker(T) = L$ so $L = \text{span}(\vec{n})$ so $\dim(L) = 1$.

$\dim(\text{im}(T)) + \dim(\ker(T)) = \underbrace{3}_{\text{dimension of the source } \mathbb{R}^3.}$

Theorem: (Rank-Nullity) Let A be an $n \times m$ matrix. Then:

$$\dim(\text{im}(A)) + \dim(\ker(A)) = m.$$

We call $\dim(\ker(A))$ the <u>nullity</u> of A. Then:

$$(\text{rank } A) + (\text{nullity of } A) = m$$

Theorem: Let A be an $n \times m$ matrix. Then:

(i) To construct a basis of $\text{im}(A)$, pick the columns of A corresponding to the columns of rref(A) having leading ones.

(ii) To construct a basis of $\ker(A)$, we can use the columns of A corresponding to the columns of rref(A) having no leading ones.

Example: $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}$ with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$; $\quad \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

columns with leading ones

$\text{im}(A)$: has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$.

$\ker(A)$: we use columns 2, 3, 5 in A. $\quad \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

columns without leading ones

$c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3 + c_4 \cdot \vec{v}_4 + c_5 \cdot \vec{v}_5 = 0$

$A\vec{x} = \vec{0}$

$\vec{v}_2 = 2 \cdot \vec{v}_1$ so $-2\vec{v}_1 + \vec{v}_2 = 0 \rightsquigarrow \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in $\ker(A)$

$\vec{v}_3 = 0 \cdot \vec{v}_1$ so $\vec{v}_3 = 0 \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ in $\ker(A)$

$\vec{v}_5 = 1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_4$ so $-\vec{v}_1 - \vec{v}_4 + \vec{v}_5 = 0 \rightsquigarrow \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ in $\ker(A)$

So $\ker(A)$ will have basis $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

so $\ker(A) = \text{span}\left(\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \right)$.

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$\ker(T) \quad \text{im}(T)$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 = V$

$\{ \ \ldots \ \}$ set