Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a basis of a subspace V in $\mathbb{R}^m$

$$\vec{x} = \underbrace{c_1}\vec{v}_1 + \dots + \underbrace{c_m}\cdot\vec{v}_m \qquad \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = [\vec{x}]_{\mathcal{B}}$$

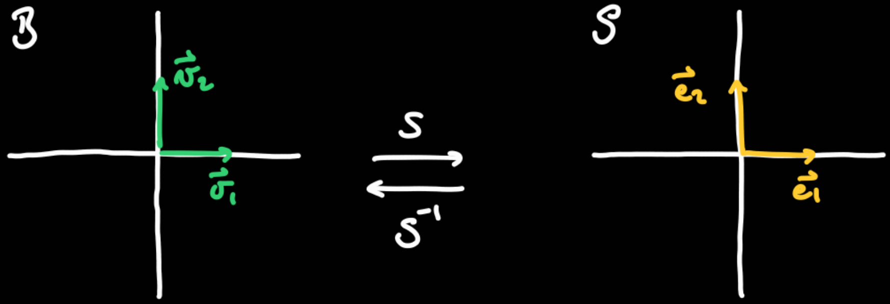
$\mathcal{B}$-coordinates of $\vec{x}$ $\qquad$ $\mathcal{B}$-coordinate vector of $\vec{x}$

$$\underbrace{x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_m \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\text{standard basis } \vec{e}_1, \dots, \vec{e}_m \quad \mathcal{S}} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \vec{x} = c_1 \cdot \vec{v}_1 + \dots + c_m \cdot \vec{v}_m =$$

$$= \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}}_{S} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = S\,[\vec{x}]_{\mathcal{B}}$$

change of basis matrix

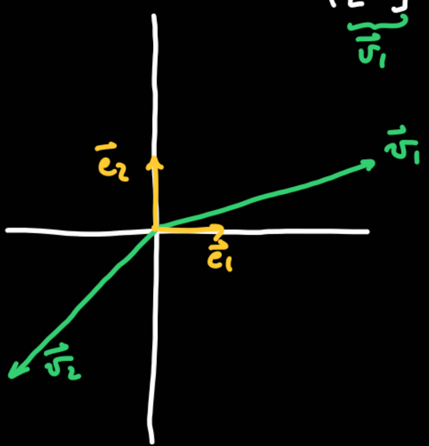
S intakes a vector in basis $\mathcal{B}$ and outputs a vector in the standard basis.



$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}} = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{B}} = 0 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2$$

Example: $\mathbb{R}^2$ $\qquad \mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} -2 \\ -2 \end{bmatrix}}_{\vec{v}_2} \right\}$

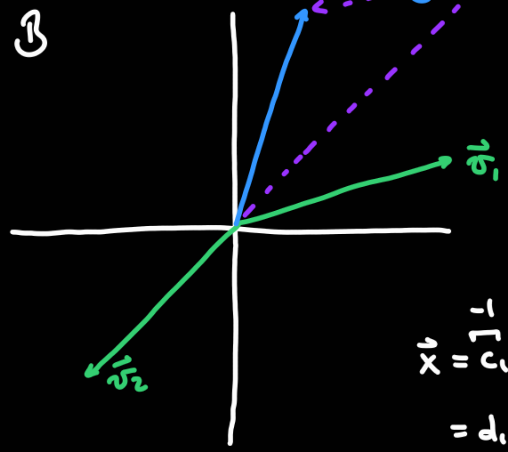


$$S = \begin{bmatrix} 3 & -2 \\ 1 & -2 \end{bmatrix}$$ takes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}}$ to $\vec{v}_1$

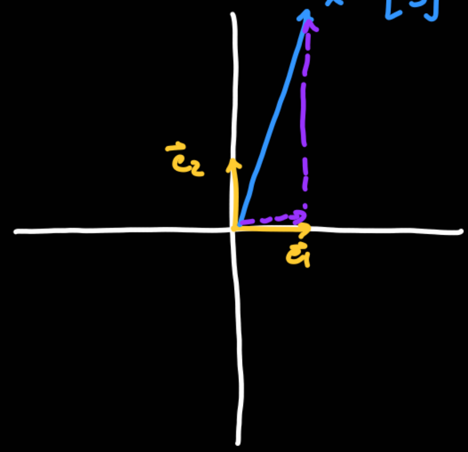
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{B}}$ to $\vec{v}_2$

$$S: \mathbb{R}^2_{\mathcal{B}} \to \mathbb{R}^2_{\mathcal{S}}$$

$\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{\mathcal{S}}$ $\qquad$ $\vec{x} = [\,\dots$



$\xrightarrow{S}$



$$\vec x = \overbrace{c_1}^{-1}\cdot\vec v_1 + \overbrace{c_2}^{-2}\cdot\vec v_2$$
$$= \underbrace{d_1}_{1}\cdot\vec e_1 + \underbrace{d_2}_{3}\cdot\vec e_2$$

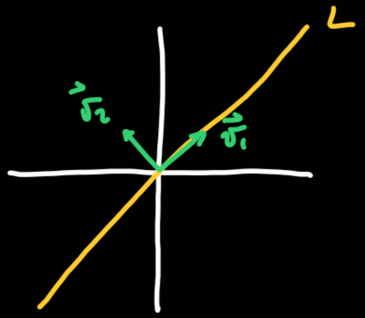
$$\vec x = \begin{bmatrix}1\\3\end{bmatrix} = -\begin{bmatrix}3\\1\end{bmatrix} - 2\cdot\begin{bmatrix}-2\\-2\end{bmatrix} = -\vec v_1 - 2\vec v_2 = \begin{bmatrix}-1\\-2\end{bmatrix}_{\mathcal{B}}$$

$$\underbrace{\begin{bmatrix}3&-2\\1&-2\end{bmatrix}}_{S}\underbrace{\begin{bmatrix}-1\\-2\end{bmatrix}}_{\mathcal{B}} = \underbrace{\begin{bmatrix}1\\3\end{bmatrix}}_{S}$$

$$\vec v_1 = \begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}} \qquad S\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\vec v_1 \text{ in } \mathcal{B}} = \underbrace{\begin{bmatrix}3\\1\end{bmatrix}}_{\vec v_1 \text{ in } S}$$

$$\begin{bmatrix}-1\\-2\end{bmatrix} = S^{-1}\begin{bmatrix}1\\3\end{bmatrix}$$

<u>Example:</u> $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ projection onto a line L.



$\mathcal{B} = \{\vec v_1, \vec v_2\}$ is a basis of $\mathbb{R}^2$.

How does T look (as a matrix) in the basis $\mathcal{B}$?

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}[T(\vec v_1)]_{\mathcal{B}} & [T(\vec v_2)]_{\mathcal{B}}\end{bmatrix}$$

$$\vec x = \underbrace{c_1\cdot\vec v_1}_{\parallel} + \underbrace{c_2\cdot\vec v_2}_{\perp}$$

$$\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}} = 1\cdot\vec v_1 + 0\cdot\vec v_2$$

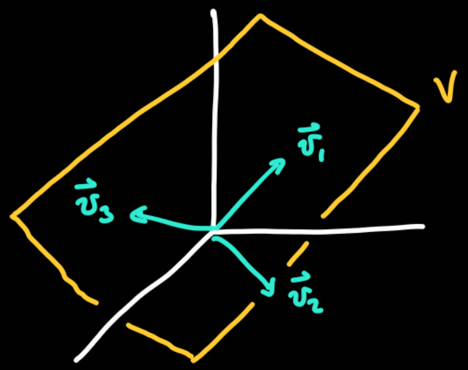
<u>Theorem:</u> Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation, $\mathcal{B} = \{\vec v_1, \ldots, \vec v_n\}$ a basis of $\mathbb{R}^n$. Then there is a unique matrix B transforms $[\vec x]_{\mathcal{B}}$ to $[T(\vec x)]_{\mathcal{B}}$: $\quad [T(\vec x)]_{\mathcal{B}} = B\,[\vec x]_{\mathcal{B}}$

$$B = \begin{bmatrix} | & & | \\ [T(\vec v_1)]_{\mathcal{B}} & \cdots & [T(\vec v_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} \qquad \begin{bmatrix} | & & | \\ T(\vec e_1) & \cdots & T(\vec e_n) \\ | & & | \end{bmatrix}$$

in any basis $\mathcal{B}$ — in the standard basis

Example: $\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}1\\0\\1\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}0\\1\\1\end{bmatrix}} \right\}$

$T: \mathbb{R}^3 \to \mathbb{R}^3$ the linear transformation projecting onto the plane $V = \underbrace{\operatorname{span}(\vec{v}_1, \vec{v}_3)}_{\text{we can find } \vec{v} \text{ from this}}$



$\vec{v} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$ normal vector to V. } defines a line L.

$\vec{v} = \vec{v}_1 \times \vec{v}_3$

$\vec{v} \cdot \vec{v}_1 = 0$

$\vec{v} \cdot \vec{v}_3 = 0$

Working in $\mathcal{S}$:

$$A = \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix}$$

← matrix of T in the basis $\mathcal{S}$

$$T(\vec{x}) = \operatorname{proj}_V(\vec{x}) = \vec{x} - \underbrace{(\vec{x} \cdot \vec{u})\vec{u}}_{\operatorname{proj}_L(\vec{x})} \qquad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$T(\vec{e}_1) = \begin{bmatrix}2/3\\1/3\\-1/3\end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix} \qquad T(\vec{e}_3) = \begin{bmatrix}-1/3\\1/3\\2/3\end{bmatrix}$$

$$A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

Working in $\mathcal{B}$:

$$B = \begin{bmatrix} | & | & | \\ [T(\vec{v}_1)]_{\mathcal{B}} & [T(\vec{v}_2)]_{\mathcal{B}} & [T(\vec{v}_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix}$$

matrix of T in basis $\mathcal{B}$

$$T(\vec{v}_1) = \vec{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}_{\mathcal{B}}$$

$$T(\vec{v}_3) = \vec{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}_{\mathcal{B}}$$

$$T(\vec{v}_2) = \frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_3 = \begin{bmatrix}1/3\\0\\1/3\end{bmatrix}_{\mathcal{B}} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u})\vec{u}$$

$$B = \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

$$\begin{tikzcd}
\mathbb{R}^3, \mathcal{S} \arrow[r, "A"] \arrow[d, "S^{-1}"'] & \mathbb{R}^3, \mathcal{S} \\
\mathbb{R}^3, \mathcal{B} \arrow[r, "B"'] & \mathbb{R}^3, \mathcal{B} \arrow[u, "S"']
\end{tikzcd}$$

<u>Check:</u> $A = S\,B\,S^{-1}$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$