Recall: A vector has length $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$

$\|\vec{u}\| = 1$ ⟵ **unit vector**

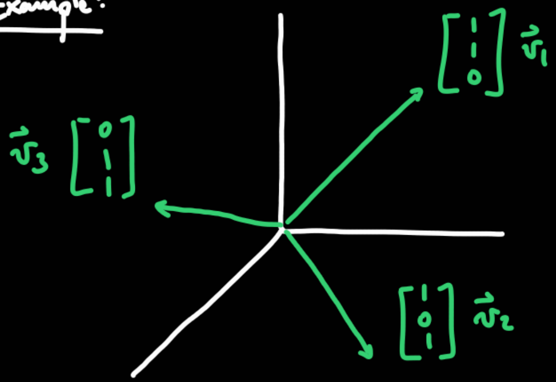
Perpendicular: $\vec{v}\cdot\vec{w} = 0$

A vector $\vec{v}$ is orthogonal to a subspace W if $\vec{v}\cdot\vec{w}=0$ for all $\vec{w}$ in W.

Remark: $\vec{v}$ is orthogonal to W if $\vec{v}$ is orthogonal to a basis of W.

$$\vec{v}\cdot\vec{w} = \vec{v}\cdot\left(c_1\cdot u_1 + \dots + c_m\cdot u_m\right) = c_1\cdot\underbrace{\vec{v}\cdot\vec{u}_1}_{0} + \dots + c_m\underbrace{\vec{v}\cdot\vec{u}_m}_{0} = 0$$

Example:



These vectors are not orthogonal.

$\|\vec{v}_i\| = \sqrt{1+1} = \sqrt{2}$

$\vec{v}_i\cdot\vec{v}_j = 1$

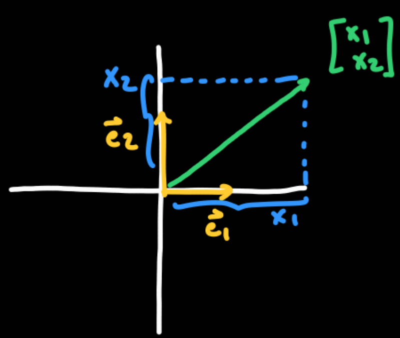
Example: Given a plane $ax+by+cz=0$, why is $\vec{x} = \begin{bmatrix}a\\b\\c\end{bmatrix}$ orthogonal to the plane?

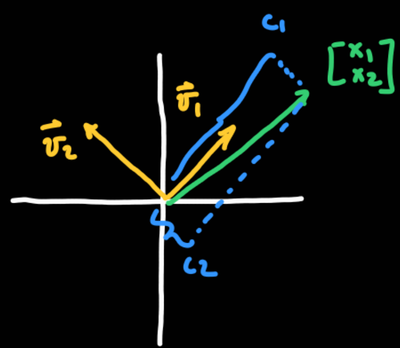
Because if $\vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ is in the plane, then:

$$\vec{x}\cdot\vec{v} = \begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \overbrace{av_1 + bv_2 + cv_3 = 0}^{av_1+bv_2+cv_3=0}$$

Motivation:

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\cdot\vec{e}_1 + x_2\cdot\vec{e}_2$$





$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1\cdot\vec{v}_1 + c_2\cdot\vec{v}_2$$

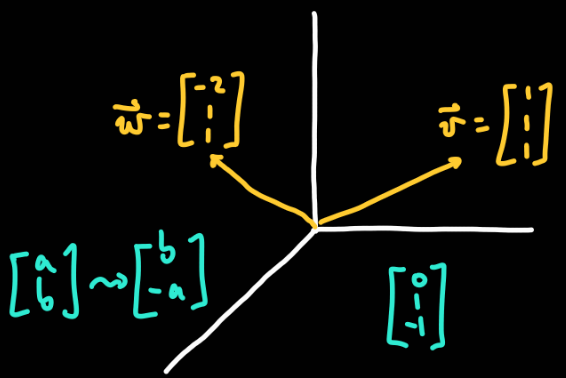
The vectors $\vec{v}_1, \dots, \vec{v}_m$ are <u>orthonormal</u> if they all have length one, and they are all perpendicular to each other.

<u>Theorem</u>:

(i) orthonormal vectors are linearly independent.

(ii) If $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$ are orthonormal, then $\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$.

<u>Example</u>: We can make any two orthogonal vectors into an orthonormal basis:



$\vec{w} \cdot \vec{v} = -2 + 1 + 1 = 0$

$\vec{u} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$ is a vector perpendicular to both

So $\vec{w}$ and $\vec{v}$ are in the plane $3y - 3z = 0$.

$\vec{v}\,' = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{w}\,' = \dfrac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ form an orthonormal basis

$\vec{u}\,' = \dfrac{\vec{u}}{\|\vec{u}\|} = \dfrac{1}{3\sqrt{2}} \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ of the plane $y - z = 0$.

Now $\mathfrak{B} = \{\vec{u}\,', \vec{v}\,', \vec{w}\,'\}$ is an orthonormal basis of $\mathbb{R}^3$.

<u>Key idea</u>: projections enable us to compute/ find perpendicular vectors

V a subspace: $\quad \vec{x} = \underbrace{\vec{x}^{\parallel}}_{\text{in } V} + \underbrace{\vec{x}^{\perp}}_{\text{perpendicular to } V}$

If $\vec{u}_1, \dots, \vec{u}_m$ an orthonormal basis of V, then:

$$\vec{x}^{\parallel} = \text{proj}_V(\vec{x}) = \underbrace{(\vec{x}\cdot\vec{u}_1)}_{c_1}\vec{u}_1 + \cdots + \underbrace{(\vec{x}\cdot\vec{u}_m)}_{c_m}\vec{u}_m$$

<u>Example:</u> V is $y - z = 0$, $\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}$ form an orthonormal basis of V. Pick $\vec{x} = \begin{bmatrix}4\\5\\6\end{bmatrix}$:

$$\vec{x}^{\parallel} = \text{proj}_V(\vec{x}) = \Bigg(\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}}_{\vec{x}\cdot\vec{u}_1}\Bigg)\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\vec{u}_1} + \Bigg(\underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix}-2 & 1 & 1\end{bmatrix}\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}}_{\vec{x}\cdot\vec{u}_2}\Bigg)\underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}}_{\vec{u}_2} =$$

$$= \begin{bmatrix}5\\5\\5\end{bmatrix} + \begin{bmatrix}-1\\1/2\\1/2\end{bmatrix} = \underbrace{\begin{bmatrix}4\\11/2\\11/2\end{bmatrix}}_{\text{in } V} \qquad 4\cdot 0 + \frac{11}{2} - \frac{11}{2} = 0$$

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \begin{bmatrix}4\\5\\6\end{bmatrix} - \begin{bmatrix}4\\11/2\\11/2\end{bmatrix} = \begin{bmatrix}0\\-1/2\\1/2\end{bmatrix}$$ ← lies in the line spanned by $\begin{bmatrix}0\\3\\-3\end{bmatrix}$, which was perpendicular to V.

$$\vec{x} = \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}4\\11/2\\11/2\end{bmatrix} + \begin{bmatrix}0\\-1/2\\1/2\end{bmatrix} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$