<u>Recall:</u> If $\underbrace{\vec{u}_1, \dots, \vec{u}_n}_{\mathfrak{B}}$ is an orthonormal basis of $\mathbb{R}^n$, then:

$$\vec{x} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{x}\cdot\vec{u}_n)\vec{u}_n \qquad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} \vec{x}\cdot\vec{u}_1 \\ \vdots \\ \vec{x}\cdot\vec{u}_n \end{bmatrix}$$

Let $V$ be a subspace of $\mathbb{R}^n$, the <u>orthogonal complement</u> $V^\perp$ of $V$ is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to all vectors in $V$.

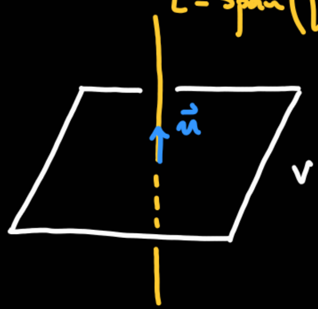
$$V^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}\cdot\vec{v} = 0 \text{ for all } \vec{v} \in V\}.$$

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto $V$, any vector $\vec{x}$ that is orthogonal to $V$ will be sent to zero: $T(\vec{x}) = \vec{0}$. The converse is also true.

So $V^\perp = \ker(T) = \ker(\text{proj}_V)$.

<u>Example:</u> Consider $T: \mathbb{R}^3 \to \mathbb{R}^3$ the linear transformation that projects any vector orthogonally onto the plane $V = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$. The associated matrix to this transformation is: $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$.

$L = \text{span}\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$




$A\vec{x} = \vec{0}$

$\vec{v} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

$$\sim \left[\begin{array}{ccc|c} 2/3 & 1/3 & -1/3 & 0 \\ 1/3 & 2/3 & 1/3 & 0 \\ -1/3 & 1/3 & 2/3 & 0 \end{array}\right]$$

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{l} x - z = 0 \\ y + z = 0 \\ z = t \end{array}$$

$$\vec{x} = \begin{bmatrix}t\\-t\\t\end{bmatrix} = t\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$\vec{e}_1\cdot\vec{u} = \frac{1}{\sqrt{3}}$

$(\vec{e}_1\cdot\vec{u})\vec{u} = \frac{1}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix}$

$\ker(T) = \text{span}\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = L = V^\perp$

$\vec{e}_1 - (\vec{e}_1\cdot\vec{u})\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}2/3\\1/3\\\end{bmatrix}$

$\ker(T) = \text{span}\left(\begin{bmatrix} \vdots \\ 1 \end{bmatrix}\right) \qquad L = V$

$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} \qquad T(\vec{x}) = \vec{x} - \vec{x}^{\perp} \qquad \vec{x}^{\perp} = \text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$

<u>Theorem:</u> Let $V$ be a subspace of $\mathbb{R}^n$, then:

(i) The orthogonal complement $V^{\perp}$ is a subspace of $\mathbb{R}^n$.

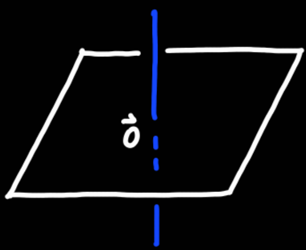
$V^{\perp} = \ker(\text{proj}_V)$

ker and im are always subspaces

(ii) The intersection of $V$ and $V^{\perp}$ is exactly $\vec{0}$.

$\vec{v} \qquad \vec{v} \qquad \|\vec{v}\| = \sqrt{\underbrace{\vec{v} \cdot \vec{v}}} = 0$

$\vec{x} \cdot \vec{v} = 0$ for all $\vec{v}$ in $V$

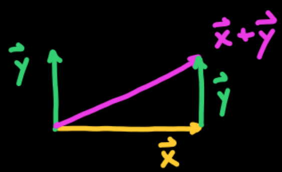



(iii) $\dim(V) + \dim(V^{\perp}) = n$ $\leftarrow$ Rank-Nullity

(iv) $(V^{\perp})^{\perp} = V$

<u>Theorem:</u> Let $\vec{x}, \vec{y}$ be vectors in $\mathbb{R}^n$, let $\theta$ be the angle between them, let $V$ be a subspace. Then:

(i) $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ if and only if $\vec{x}$ and $\vec{y}$ are orthogonal.




(ii) $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$ with equality if and only if $\vec{x}$ is in $V$.

(iii) (Cauchy-Schwartz) $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \cdot \|\vec{y}\|$

with equality if and only if $\vec{x}$ is parallel to $\vec{y}$.

(iv) $\cos(\theta) = \dfrac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\cdot\|\vec{y}\|}$.

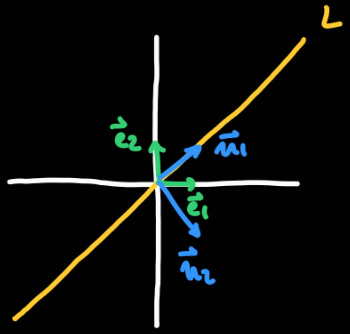
$|\cos(\theta)| \leq 1$

<u>Example:</u> $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$. This is <u>not</u> an orthonormal basis:

(i) These vectors have length $\sqrt{2}$.

(ii) The dot product of any two vectors is 1, so:

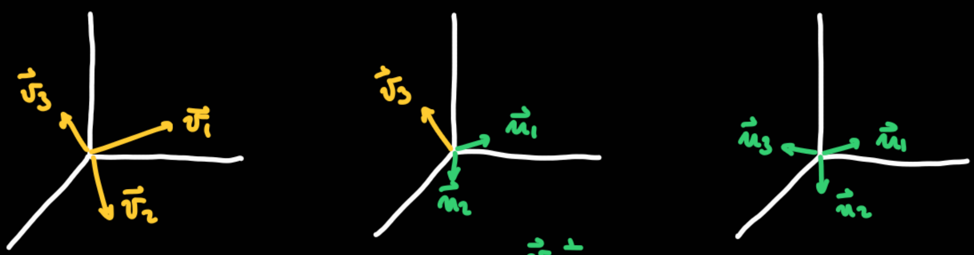
$$\cos(\theta) = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2} \quad \text{so} \quad \theta = \frac{\pi}{3}, \text{ which is } \underline{\text{not}}\ \frac{\pi}{2}.$$




$$\mathrm{ref}_L(\vec{x}) = \vec{x} - 2\vec{x}^{\perp} = \vec{x}^{\parallel} - \vec{x}^{\perp}$$

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp} \xrightarrow{T} T(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$$

$T(\vec{u}_1) = \vec{u}_1$ , $T(\vec{u}_2) = -\vec{u}_2$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$




$\vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp}$ $\qquad \vec{u}_2 = \dfrac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|}$

with respect to $\vec{v}_1$