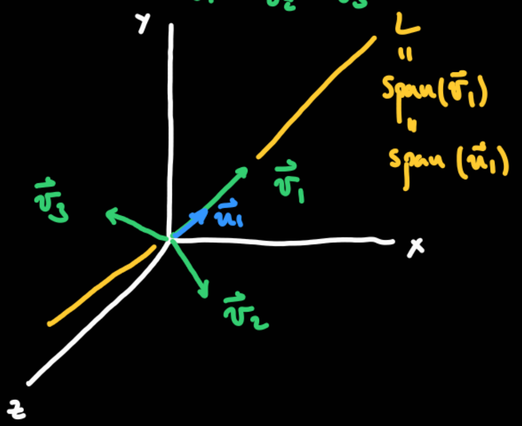
Example: $\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}1\\0\\1\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}0\\1\\1\end{bmatrix}} \right\}$ is a basis of $\mathbb{R}^3$



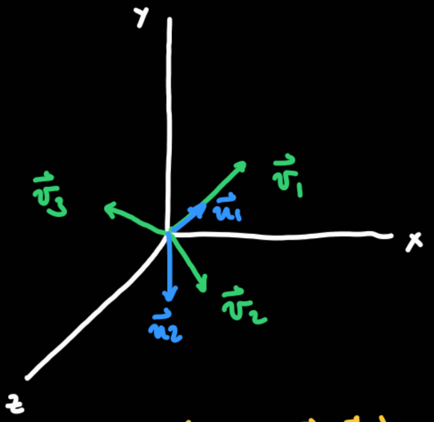
$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp}$$

$$\vec{v}_2^{\parallel} = \mathrm{proj}_L(\vec{v}_2) = (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1 =$$

$$= \left(\begin{bmatrix}1 & 0 & 1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \right) \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}.$$



$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{2}{\sqrt{6}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

$V = \mathrm{span}(\vec{v}_1, \vec{v}_2) = \mathrm{span}(\vec{u}_1, \vec{u}_2)$ $\quad$ $\vec{v} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$ is perpendicular to V

1. $\vec{u}_1 \times \vec{u}_2$

$c_1 = (\vec{v}_3 \cdot \vec{u}_1) \quad c_2 = (\vec{v}_3 \cdot \vec{u}_2)$

2. Project $\vec{v}_3$ into V to find $\vec{v}_3^{\parallel} = c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2$, and then subtract

$\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{\parallel}$. Then $\vec{u}_3 = \dfrac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|}$.

$$\vec{v}_3^{\perp} = \left(\vec{v}_3 \cdot \frac{\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|} = \left(\begin{bmatrix}0 & 1 & 1\end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix} \right) \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix} = \frac{-2}{3}\begin{bmatrix}1\\-1\\-1\end{bmatrix} = \frac{2}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

$$\vec{v}_3^{\perp} = \mathrm{proj}_{V^{\perp}}(\vec{v}_3)$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{1}{\sqrt{3}} \cdot \frac{3}{2} \cdot \frac{2}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\mathcal{A} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix} \right\}$$ is an orthonormal basis of $\mathbb{R}^3$.

Gram-Schmidt process:

$\{\vec{v}_1, \dots, \vec{v}_m\}$ a basis of V

Decompose $\vec{v}_j = \vec{v}_j^{\parallel} + \vec{v}_j^{\perp}$ with respect to $\vec{v}_1, \dots, \vec{v}_{j-1}$.

$\vec{v}_j^{\parallel}$: in $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_{j-1})$ $\quad$ $\vec{v}_j^{\perp}$: perpendicular to $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_{j-1})$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \quad \vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|}, \quad \dots, \quad \vec{u}_m = \frac{\vec{v}_m^{\perp}}{\|\vec{v}_m^{\perp}\|}$$

these vectors form an orthonormal basis of V.

$$\mathcal{B} \; \{\vec{v}_1, \dots, \vec{v}_m\} \rightsquigarrow \{\vec{u}_1, \dots, \vec{u}_m\} \; \mathcal{A}$$

$$S_{\mathcal{B}} = M \xleftrightarrow{\quad R \quad} S_{\mathcal{A}} = Q$$

$M = QR$

$Q = M \cdot R^{-1}$

$$\mathcal{B} \xrightarrow[\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}]{} \mathcal{S}$$

QR factorization:

M $\quad n \times m$ with linearly independent columns

$$\overset{n\times m}{M} = \overset{n\times m}{Q}\;\overset{m\times m}{R}$$

Q: has orthonormal columns

R: upper triangular matrix with strictly positive diagonal entries

Is R invertible? Yes.

Example: $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$ $\qquad M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \end{bmatrix}$

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}, \quad \vec{u}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix} \qquad Q = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13}\\ 0 & r_{22} & r_{23}\\ 0 & 0 & r_{33}\end{bmatrix}$$

$$r_{11} = \vec{u}_1 \cdot \vec{v}_1 = \sqrt{2} \qquad r_{12} = \vec{u}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}}$$

$$r_{13} = \vec{u}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}} \qquad r_{22} = \vec{u}_2 \cdot \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \frac{3}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$r_{23} = \vec{u}_2 \cdot \vec{v}_3 = \frac{1}{\sqrt{6}} \qquad r_{33} = \frac{2}{\sqrt{3}}$$

$$R = \begin{bmatrix}\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix} \qquad M = QR$$

$$QR = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} = M$$