Denote $M_n(\mathbb{R})$ the set of all $n\times n$ matrices, a <u>determinant</u> is a function

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R} \quad \text{satisfying:}$$

(i) It is linear with respect to columns:

$$\det\begin{bmatrix} | & & | & | & & | \\ \vec{c}_1, & \dots, & \vec{c}_i + & \vec{c}_i{}', & \dots, & \vec{c}_n \\ | & & | & | & & | \end{bmatrix} = \det\begin{bmatrix} | & & | & & | \\ \vec{c}_1, & \dots, & \vec{c}_i, & \dots, & \vec{c}_n \\ | & & | & & | \end{bmatrix} + \det\begin{bmatrix} | & & | & & | \\ \vec{c}_1, & \dots, & \vec{c}_i{}', & \dots, & \vec{c}_n \\ | & & | & & | \end{bmatrix}$$

$$2\cdot\det\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \det\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \det\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} + \det\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$2+2 \qquad 2\cdot 2$

$1+1 \qquad 2\cdot 1$

$$\det\begin{bmatrix} | & & | & & | \\ \vec{c}_1, & \dots, & k\,\vec{c}_i, & \dots, & \vec{c}_n \\ | & & | & & | \end{bmatrix} = k\cdot\det\begin{bmatrix} | & & | & & | \\ \vec{c}_1, & \dots, & \vec{c}_i, & \dots, & \vec{c}_n \\ | & & | & & | \end{bmatrix}$$

(ii) It is alternating with respect to columns:

$$\det\begin{bmatrix} | & & | & & | & & | \\ \vec{c}_1, & \dots, & \vec{c}_i, & \dots, & \vec{c}_i, & \dots, & \vec{c}_n \\ | & & | & & | & & | \end{bmatrix} = 0.$$

(iii) The determinant of the identity is 1:

$$\det\begin{bmatrix} | & & | \\ \vec{e}_1, & \dots, & \vec{e}_n \\ | & & | \end{bmatrix} = 1.$$

<u>Example:</u> $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$

$$\det\begin{bmatrix} a+a' & b \\ c+c' & d \end{bmatrix} = (a+a')d - (c+c')b = ad - bc + a'd - bc' =$$

$$= \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det\begin{bmatrix} a' & b \\ c' & d \end{bmatrix}$$

<u>Example:</u>

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$

$$\det(A) = a_{11} \cdot a_{22} \cdots a_{n-1n-1} \cdot a_{nn}$$

Theorem: Let A be an invertible matrix, if when computing $\mathrm{rref}(A)$ we swap rows s times and we divide rows by the scalars $k_1, \dots, k_r$ then:

$$\det(A) = (-1)^s \cdot k_1 \cdots k_r$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow{\frac{1}{-4} \cdot R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{bmatrix}$$

$$\xrightarrow[R_3+3R_2]{R_1-2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2} R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2-2R_1]{R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = -4 \cdot 2 = -8$$

$$\det(A) = 1 \cdot 2 \cdot 2 + 2 \cdot 1 \cdot 2 + 3 \cdot 3 \cdot 1 - 3 \cdot 2 \cdot 2 - 1 \cdot 1 \cdot 1 - 2 \cdot 3 \cdot 2 = -8$$

Let A be an $n \times n$ matrix, the $(n-1) \times (n-1)$ matrix A_{ij} obtained by removing the i-th row and j-th column of A is called a <u>submatrix</u>. The determinant of A_{ij}

is called a <u>minor</u> of A.

<u>Theorem:</u>

Expansion by columns: $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$

Expansion by rows: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$

<u>Example:</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

1^{st} col. $\det(A) = (-1)^{1+1} \cdot 1 \cdot \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + (-1)^{2+1} \cdot 3 \cdot \det \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + (-1)^{3+1} \cdot 2 \cdot \det \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} =$

$= 1 \cdot 3 - 3 \cdot 1 + 2 \cdot (-4) = -8$

2^{nd} col. $\det(A) = (-1)^{1+2} \cdot 2 \cdot \det \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} + (-1)^{2+2} \cdot 2 \cdot \det \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} + (-1)^{3+2} \cdot 1 \cdot \det \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} =$

$= -2 \cdot 4 + 2 \cdot (-4) - 1 \cdot (-8) = -8$

$A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$\det(A) = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$