Recall: $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij}) \qquad j\text{-th column}$$

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij}) \qquad i\text{-th row}$$

This means that the determinant is symmetric with respect to rows and columns.

$A \qquad A^T = A \qquad \det(A^T) = \det(A)$

(i) The determinant is linear with respect to each row:

$$\det\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i + \vec{r}_i{}' & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} = \det\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} + \det\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i{}' & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix}$$

$$\det\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & k\vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} = k\det\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix}$$

$$\det\begin{bmatrix} 2 & 2 & 2 \\ 4 & 6 & 4 \\ 3 & 1 & 2 \end{bmatrix} = 2 \cdot \det\begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} = 2 \cdot 2 \cdot \det\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

(ii) The determinant is alternating in the rows:

$$\det\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 4 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix}$$

(iii) The determinant of the identity matrix is 1:

$$\det\begin{bmatrix} - & \vec{e}_1 & - \\ & \vdots & \\ - & \vec{e}_n & - \end{bmatrix} = 1$$

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -2$$

$$\det\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = +2 = -\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Challenge:

$$\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} \begin{matrix} \leftarrow i \\ \leftarrow j \end{matrix}$$

$$\det\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \end{bmatrix} = -\det\begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \end{bmatrix}$$

$$\det\begin{bmatrix} \vdots \\ - \vec{r}_n - \end{bmatrix} = -\det\begin{bmatrix} - \vec{r}_j - \\ \vdots \\ - \vec{r}_i - \\ \vdots \\ - \vec{r}_n - \end{bmatrix}$$ swapped i and j rows

<u>Hint:</u> Use the alternating property.

Because of this symmetry, $\det(A^T) = \det(A)$.

<u>Example:</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ $\quad \det(A) = -8$

$$\det(A^T) = (-1)^{1+1}\cdot 1\cdot \det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + (-1)^{2+1}\cdot 2\cdot\det\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} + (-1)^{3+1}\cdot 3\cdot\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} =$$

$$= 1\cdot 3 - 2\cdot 4 + 3\cdot(-1) = -8 = \det(A).$$

It is useful to remember how elementary row operations affect the determinant:

(i) If B is obtained from A by multiplying a row of A by k, then:
dividing

$\det(B) = k\cdot\det(A).$
$\frac{1}{k}$

$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \xrightarrow{3\cdot R_2} B = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$ $\quad \det(A) = -2$

$\det(B) = 12 - 18 =$
$= -6 = 3\cdot\det(A)$

(ii) If B is obtained from A by swapping two rows, then:

$\det(B) = -\det(A)$

$\Big(f(x,y) = x - y, \quad g(x,y) = y - x =$
f is <u>alternating</u> $\qquad = -(x-y) =$
$= -f(x,y) \Big)$

(iii) If B is obtained from A by adding a multiple of a row of A to another row of A, then:

$\det(B) = \det(A).$

$A = \begin{bmatrix} 5 & 3 \\ \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 4 \\ \end{bmatrix}$

$4 \quad 1 \qquad\qquad 4 \quad 1$

$\det(A) = -7 \qquad \det(B) = -7$

$$\det\begin{bmatrix} 9 & 4 \\ 4 & 1 \end{bmatrix} = \det\begin{bmatrix} 5+4 & 3+1 \\ 4 & 1 \end{bmatrix} = \det\begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix} + \det\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} = \det\begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix}$$

Theorem: A matrix is invertible if and only if its determinant is <u>not</u> zero.

$$\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$$[A \mid I_n] \xrightarrow[\text{multiplication}]{\text{swaps}} [I_n \mid A^{-1}]$$

$$A^{-1} \xrightarrow[\text{divide}]{\text{swaps}} I_n$$

$$A \longrightarrow I_n \qquad \det(A) = -8$$

$$I_n \longrightarrow A^{-1}$$

WARNING: $\det(A+B) \neq \det(A) + \det(B)$

$$\det(I_n - I_n) \stackrel{?}{=} \det(I_n) + \det(-I_n)$$

$$\det(kA) \neq k \cdot \det(A)$$

Theorem:

(i) $\det(kA) = k^n \cdot \det(A)$ $\quad A$ is $n \times n$

(ii) $\det(AB) = \det(A)\det(B)$

(iii) $\det(A^m) = \det(A)^m$

(iv) If A and B are similar then $\det(A) = \det(B)$.

$AS = SB$

$$\det(AS) = \det(SB)$$

$$\det(A)\cdot\det(S) \qquad \det(S)\cdot\det(B)$$