Question: What is the determinant of an orthogonal matrix?

Answer: ± 1.

A orthogonal means $A^T A = I_n$. So:

$$\det(A^T A) = \det(I_n) \longrightarrow \det(A^T)\det(A) = 1$$

$$\longrightarrow \det(A)\det(A) = 1 \longrightarrow (\det(A))^2 = 1$$

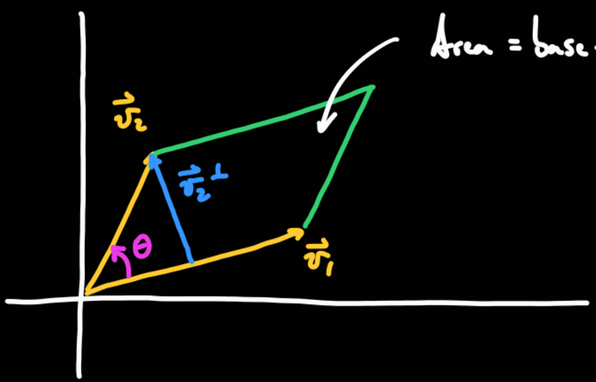
So $\det(A) = \pm 1$. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is orthogonal and has determinant -1.

Geometric interpretation of the determinant:

$A = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$ a 2×2 matrix, $\det(A) = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \sin(\theta)$ where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$.

$A = \begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{bmatrix}$ $\det(A) = \vec{w}_1 \cdot (\vec{w}_2 \times \vec{w}_3)$

$\det(A) = \|\vec{w}_1\| \cdot \|\vec{w}_2^{\perp}\| \cdot \|\vec{w}_3^{\perp}\|$



Area = base $\cdot$ height $= \|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\|$

$\|\vec{v}_2^{\perp}\| = \|\vec{v}_2\| \cdot |\sin(\theta)|$

Thus $|\det(A)| = \|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\|$ is the area of the parallelogram spanned by $\vec{v}_1$ and $\vec{v}_2$.

Let A be any invertible matrix, $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$, using the Gram-Schmidt process we can find a QR decomposition of A:

upper triangular with strictly positive diagonal entries

$A = QR$

$$A = \underbrace{Q}_{\text{orthogonal}} R$$

$$|\det(A)| = |\det(QR)| = |\det(Q)\det(R)| = \underbrace{|\det(Q)|}_{1}\cdot|\det(R)| = r_{11}\cdot r_{22}\cdots r_{nn} =$$

$$= \|\vec{v}_1\|\cdot\|\vec{v}_2^{\perp}\|\cdots\|\vec{v}_n^{\perp}\|.$$

$$r_{11} = \|\vec{v}_1\|,\ r_{22} = \|\vec{v}_2^{\perp}\|,\ \ldots,\ r_{nn} = \|\vec{v}_n^{\perp}\|$$

$\vec{v}_i^{\perp}$ is the component of $\vec{v}_i$ perpendicular to $\operatorname{span}(\vec{v}_1,\ldots,\vec{v}_{i-1})$.

This formula exactly generalizes the formula :

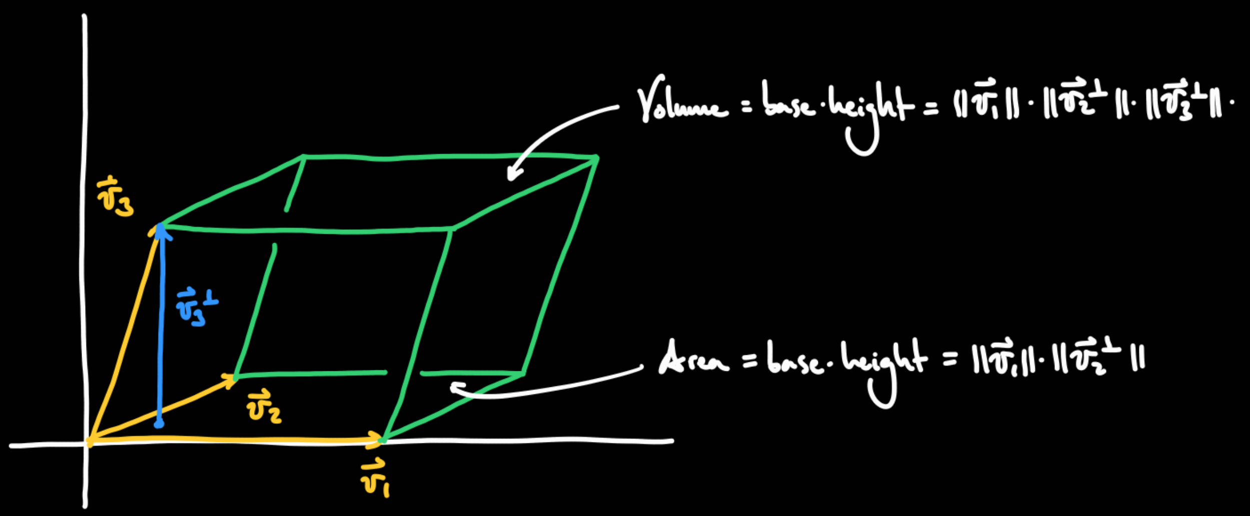
Area = base · height

to any dimension n.

"length" is the 1-volume $\qquad \mathbb{R}^n$

"area" is the 2-volume

"volume" is the 3-volume



The volume in $\mathbb{R}^n$ of the linearly independent vectors $\vec{v}_1,\ldots,\vec{v}_n$ is $\left|\det\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}\right|$.

Theorem: (Cramer's rule) Let $A\vec{x} = \vec{b}$ be a linear system, A is invertible, then the solution has in the i-th entry:

$$x_i = \frac{\det(A_{\vec{b},i})}{\det(A)}$$

where $A_{\vec{b},i}$ is obtained from A by replacing the i-th column with $\vec{b}$.

The **classical adjoint** of an invertible matrix A is: (denoted $\operatorname{adj}(A)$)

$$(\operatorname{adj}(A))_{ij} = (-1)^{i+j} \cdot \det(A_{ji}).$$

Theorem: $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$. To compute A^{-1} we first find all the minors $\det(A_{ij})$ and put them in a 3x3 matrix in their respective positions with their respective signs:

$$\begin{bmatrix} \overset{+}{(-1)^{1+1}} \det(A_{11}) & \overset{-}{(-1)^{1+2}} \det(A_{12}) & \overset{+}{(-1)^{1+3}} \det(A_{13}) \\ \overset{-}{(-1)^{2+1}} \det(A_{21}) & \overset{+}{(-1)^{2+2}} \det(A_{22}) & \overset{-}{(-1)^{2+3}} \det(A_{23}) \\ \overset{+}{(-1)^{3+1}} \det(A_{31}) & \overset{-}{(-1)^{3+2}} \det(A_{32}) & \overset{+}{(-1)^{3+3}} \det(A_{33}) \end{bmatrix} = \begin{bmatrix} 3 & -4 & -1 \\ -1 & -4 & 3 \\ -4 & 8 & -4 \end{bmatrix}$$

now we take the transpose:

$$\begin{bmatrix} 3 & -1 & -4 \\ -4 & -4 & 8 \\ -1 & 3 & -4 \end{bmatrix} = \operatorname{adj}(A)$$

finally, we divide by $\det(A) = -8$:

A^{-1} $\operatorname{adj}(A)$ $\begin{bmatrix} -3/8 & 1/8 & 1/2 \end{bmatrix}$

$$A = \frac{[illegible]}{\det(A)} = \begin{bmatrix} [illegible] & [illegible] & [illegible] \\ 1/2 & 1/2 & -1 \\ 1/8 & -3/8 & 1/2 \end{bmatrix}$$