5. Eigenvalues and eigenvectors.

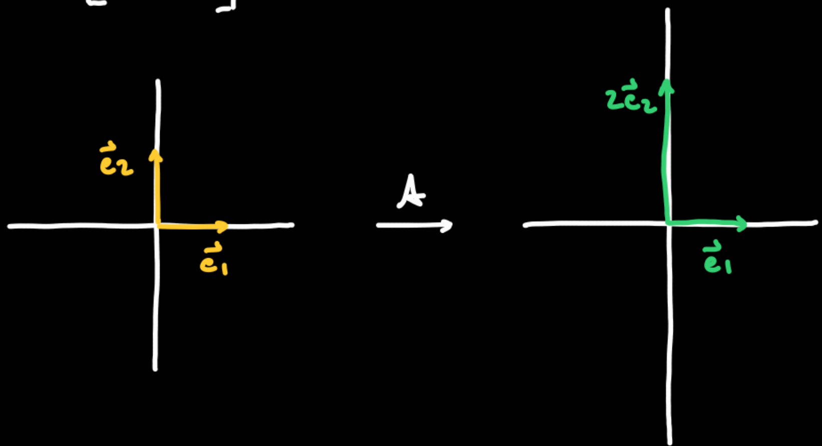
Informally, eigenvectors are "preferred directions" of a matrix, and eigenvalues are the scaling factors happening in those "preferred directions".

Examples:

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, any vector $\vec{v}$ in $\mathbb{R}^2$ is sent to itself.

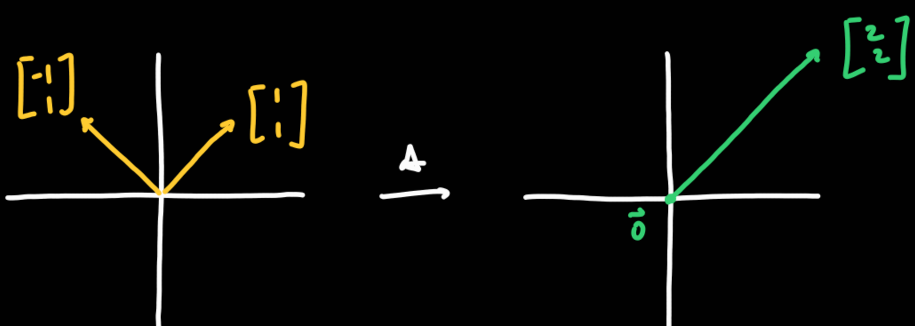
   Thus any vector $\vec{v}$ is a "preferred direction" of A, with scaling factor 1.

2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, sends $\vec{e}_1$ to $\vec{e}_1$ and $\vec{e}_2$ to $2\vec{e}_2$.



   The "preferred directions" of A are $\vec{e}_1$ and $\vec{e}_2$, with scaling factors of 1 and 2 respectively.

3. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, sends $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.



The "preferred directions" are $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$ with scaling factors of 2 and 0 respectively.

$$\begin{bmatrix}1&1\\1&1\end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}}_{\text{projection onto the line } y=x}\underbrace{\begin{bmatrix}2&0\\0&2\end{bmatrix}}_{\text{scaling factor of two}}$$

4. $A = \begin{bmatrix}2/3&1/3&-1/3\\1/3&2/3&1/3\\-1/3&1/3&2/3\end{bmatrix}$, sends $\begin{bmatrix}1\\1\\0\end{bmatrix}$ to $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$ to $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

Also $\begin{bmatrix}1\\-1\\1\end{bmatrix}$ is sent to $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

The "preferred directions" of A are $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}$ with scaling factors 1, 1, 0 respectively.

We knew this! A is the orthogonal projection onto $V = \operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$ so it will keep everything in V untouched, and the perpendicular component will be sent to zero. We know that $V^{\perp} = \operatorname{span}\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$.

<u>Example:</u>

1. $A = \begin{bmatrix}1&0\\0&2\end{bmatrix}$, $A(\vec{e}_1) = \vec{e}_1$ and $A(\vec{e}_2) = 2\vec{e}_2$. $\mathfrak{B} = \{\vec{e}_1, \vec{e}_2\}$

The matrix of the linear transformation associated to A in $\mathfrak{B}$ is:

$$B = \begin{bmatrix}[A(\vec{e}_1)]_{\mathfrak{B}} & [A(\vec{e}_2)]_{\mathfrak{B}}\end{bmatrix} = \begin{bmatrix}1 & 0 \\ \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \ & \ \\ 0 & 2 \end{bmatrix}}_{\text{diagonal}}$$

2. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\quad A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\quad A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $\quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

The matrix of the linear transformation associated to A in $\mathcal{B}$ is:

$$B = \left[\left[A\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \ \left[A\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{diagonal}}$$

3. $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $\quad A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\quad A\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\quad A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

The matrix of the linear transformation associated to A in $\mathcal{B}$ is:

$$B = \left[\left[A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \ \left[A\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \ \left[A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{diagonal}}$$

<u>Example</u>: Not all matrices have "preferred directions".

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, a counterclockwise rotation of $\frac{\pi}{2}$, does <u>not</u> have "preferred directions".

This happens because we are working over $\mathbb{R}$. If we work over $\mathbb{C}$, every matrix has "preferred directions".

Let A be an $n \times n$ matrix. A non-zero vector $\vec{v}$ in $\mathbb{R}^n$ is called an <u>eigenvector</u> of A

if: $A\vec{v} = \lambda \vec{v}$ for some real scalar λ. The scalar λ is called the <u>eigenvalue</u>

associated to the eigenvector $\vec{v}$.

<u>Example:</u>

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenvectors $\vec{e}_1$ and $\vec{e}_2$ with eigenvalues 1 and 2.
2. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with eigenvalues 2 and 0.
3. $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ with eigenvalues 1, 1, 0.
4. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ does not have eigenvectors nor eigenvalues.

<u>Theorem:</u> Let A be an $n \times n$ matrix with eigenvectors $\vec{v}_1, \dots, \vec{v}_n$. If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$
eigenvalues $\lambda_1, \dots, \lambda_n$

form a basis of $\mathbb{R}^n$, then:

$$A \text{ is similar to } B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \text{ via } S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}.$$

$$B = S^{-1} A S.$$

<u>Theorem:</u> Let A be an $n \times n$ matrix and $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$ such that

A is similar to $B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ via $S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$. Then A has

eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$.