<u>Recall:</u> A vector $\vec{v}$ (non-zero) is an <u>eigenvector</u> of a matrix A if $A\vec{v} = \underbrace{\lambda}_{\text{real scalar}}\vec{v}$.

The scalar λ is an <u>eigenvalue</u> of A.

$$A\vec{v} - \lambda\vec{v} = \vec{0} \qquad \underbrace{(A-\lambda I_n)}_{\text{this matrix can have determinant zero}}\vec{v} = \vec{0}$$

Let A be an $n\times n$ matrix, a real scalar λ is an eigenvalue of A if and only if $\det(A-\lambda I_n) = 0$. This equality is called the <u>characteristic equation</u> of A.

When we see λ as a variable, $\det(A-\lambda I_n)$ is a polynomial of degree n, called the <u>characteristic polynomial</u> of A, denoted by $f_A(\lambda)$.

<u>Example:</u> Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

A sends $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

To obtain the eigenvalues of A, we find the roots of its characteristic polynomial:

$$f_A(\lambda) = \det(A-\lambda I_2) = \det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1 = \lambda^2 + 1 - 2\lambda - 1 =$$

$$= \lambda^2 - 2\lambda = \underbrace{\lambda}_{0}\cdot\underbrace{(\lambda-2)}_{2}$$

The roots of $f_A(\lambda)$ are $\lambda = 0$, $\lambda = 2$, so the eigenvalues of A are 0 and 2.

$$(A-\lambda I_n)\vec{v} = \vec{0} \qquad (A-\lambda I_n)\vec{x} = \vec{0}$$

Let A be a square matrix, the <u>trace</u> of A is the sum of its diagonal entries, denoted by $tr(A)$.

<u>Example:</u> $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find its characteristic polynomial.

$$f_A(\lambda) = \det(A - \lambda I_2) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc =$$

$$= \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - tr(A)\lambda + \det(A).$$

Let A be an $n \times n$ matrix with eigenvalue λ_0. The <u>algebraic multiplicity</u> of λ_0 is how many times it appears in the characteristic polynomial. Equivalently, it is the largest integer k such that:

$$f_A(\lambda) = (\lambda - \lambda_0)^k \cdot g(\lambda) \quad \text{with } g(\lambda_0) \neq 0.$$

<u>Example:</u> $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, find its eigenvalues with their algebraic multiplicities.

$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\text{preferred directions}}$ are sent to $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The characteristic polynomial of A is:

$$f_A(\lambda) = \det \begin{bmatrix} 2/3 - \lambda & 1/3 & -1/3 \\ 1/3 & \frac{2}{3} - \lambda & 1/3 \\ -1/3 & \frac{1}{3} & \frac{2}{3} - \lambda \end{bmatrix} = -\lambda^3 + 2\lambda^2 - \lambda = \underbrace{-\lambda}_{\substack{0 \\ 1}} \cdot \underbrace{(\lambda - 1)^2}_{\substack{1 \\ 2}}.$$

The eigenvalues of A are 1 and 0, with algebraic multiplicities 2 and 1 respectively.

Theorem: An $n \times n$ matrix has at most n distinct real eigenvalues. If n is odd then we have at least one real eigenvalue. If n is even we may not have real eigenvalues.

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find all eigenvectors and eigenvalues.

The characteristic polynomial is:

$$f_A(\lambda) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$, which has no real roots.

Thus A has no real eigenvalues, so A does not have eigenvectors.

Theorem: Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ listed with multiplicity.

Then:

$$\det(A) = \lambda_1 \cdots \lambda_n \quad \text{and} \quad tr(A) = \lambda_1 + \cdots + \lambda_n.$$