Recall: $A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0} \iff \underbrace{(A - \lambda I_n)}\vec{v} = \vec{0}$

the eigenvectors live in the kernel of $A - \lambda I_n$.

The *eigenspace* of λ, denoted E_λ, is the kernel of $A - \lambda I_n$.

$$E_\lambda = \ker(A - \lambda I_n) = \{\vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda\vec{v}\}.$$

Example: $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, find its eigenspaces. $\underbrace{\lambda = 0 \quad \lambda = 1}_{\text{eigenvalues of } A}$

$f_A(\lambda) = -\lambda \cdot (\lambda - 1)^2$

To find E_0 we have to find $\ker(A)$, so we have to solve $A\vec{x} = \vec{0}$.

solution

$$\vec{x} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{so} \quad E_0 = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right).$$

To find E_1 we have to find $\ker(A - I_3)$, so we have to solve $(A - I_3)\vec{x} = \vec{0}$.

solution

$$\vec{x} = \begin{bmatrix} t \\ t+s \\ s \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{so} \quad E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right).$$

The *geometric multiplicity* of λ, denoted $\operatorname{gemu}(\lambda)$, is the dimension of E_λ.

$$\operatorname{gemu}(\lambda) = \dim(\ker(A - \lambda I_n)) = \operatorname{nullity}(A - \lambda I_n) = n - \operatorname{rank}(A - \lambda I_n).$$

Let A be an $n \times n$ matrix, an *eigenbasis* of A is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Example:

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenbasis $\mathfrak{B} = \{\vec{e}_1, \vec{e}_2\}$. $\quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenbasis $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$ has eigenbasis $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ does <u>not</u> have an eigenbasis.

<u>Remark:</u> Let A be an $n \times n$ matrix.

(a) Find a basis of each E_λ, put all these vectors next to each other:

$$\vec{v}_1, \dots, \vec{v}_s \quad \text{with } s \text{ the sum of the geometric multiplicities of } A.$$

(b) The vectors $\vec{v}_1, \dots, \vec{v}_s$ are linearly independent.

(c) The vectors $\vec{v}_1, \dots, \vec{v}_s$ are an eigenbase of A if and only if $s = n$.

<u>Theorem:</u> Let A be an $n \times n$ matrix with n distinct eigenvalues. Then there is an eigenbasis of A, to construct it we find one eigenvector for each eigenvalue.

<u>Theorem:</u> Let A be similar to B, then:

(a) $f_A(\lambda) = f_B(\lambda)$

(b) $\operatorname{rank}(A) = \operatorname{rank}(B)$ and $\operatorname{null}(A) = \operatorname{null}(B)$

(c) The eigenvalues, algebraic and geometric multiplicities of A and B coincide.

(d) $\det(A) = \det(B)$ and $\operatorname{tr}(A) = \operatorname{tr}(B)$.

<u>Example:</u> The algebraic and geometric multiplicities of $A = \begin{bmatrix} 8 & -9 \\ 4 & -4 \end{bmatrix}$ are

different.

$$f_A(\lambda) = (8-\lambda)(-4-\lambda) - (-9)\cdot 4 = (\lambda-2)^2 \qquad \lambda = 2 \quad \text{almu}(2) = 2$$

$$\text{gemu}(2) = 2 - \text{rank}\,(A - 2I_2) = 2 - \text{rank}\left(\begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}\right) = 2-1 = 1$$

$\text{gemu}(2) = 1.$

<u>Theorem:</u> $\text{gemu}(\lambda) \leq \text{almu}(\lambda)$.