Examples: Find an eigenbasis and a diagonal matrix similar to:

1. $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

1. $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ find solutions to $\overbrace{\det(A - \lambda I_2)}^{f_A(\lambda)} = 0$

$f_A(\lambda) = \underbrace{(1-\lambda)}_{1}\underbrace{(2-\lambda)}_{2}$ Thus A has eigenvalues $\lambda = 1$ and $\lambda = 2$.

"If we have an $n \times n$ matrix with n distinct eigenvalues then it is similar to a diagonal matrix having those eigenvalues in the diagonal."

E_1 : $\ker(A - I_2)$ $\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} \vec{x} = \vec{0}$ $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ a solution.

E_2 : $\ker(A - 2I_2)$ $\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$ $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ a solution.

$\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$, A is similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

2. $\underbrace{A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{}$. 2×2

$f_A(\lambda) = (1-\lambda)(1-\lambda) = \underbrace{(1-\lambda)^2}_{\lambda = 1}$

$\lambda = 1$ is an eigenvalue with algebraic multiplicity 2.

E_1 : $\ker(A - I_2)$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$ $\vec{x} = t \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so $E_1 = \text{span}\left(\begin{bmatrix} \\ 0 \end{bmatrix}\right)$ has dim: 1, so $\text{gemu}(1) = 1$.

The sum of the geometric multiplicities of A is 1, which is not 2, so

A does not have an eigenbasis.

3. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Answer: 1. No because it has two rows of zeroes.

This *is* diagonalizable!

$$f_A(\lambda) = \underbrace{(1-\lambda)}_{1}\underbrace{(-\lambda)}_{0}\underbrace{(-\lambda)}_{0}$$

E_0 : $\ker(A)$ $\quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{x} = \vec{0} \qquad \vec{x} = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = t\cdot\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\cdot\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

$\dim(E_0) = \text{gemu}(0) = 2 \qquad E_0 = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$

E_1 : $\ker(A - I_3)$ $\quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\vec{x} = \vec{0} \qquad \vec{x} = t\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\dim(E_1) = \text{gemu}(1) = 1 \qquad E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right).$

When the geometric multiplicities add up to the size of the matrix, there is an eigenbasis. This eigenbasis is formed by putting together the basis elements of the eigenspaces.

$\mathfrak{B} = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$, so A is similar to $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} \lambda_1 & 0 \end{bmatrix}^m$ $\qquad$ Rmk: $A = SBS^{-1}$

$$\begin{bmatrix} \ddots & \\ 0 & \lambda_n \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{bmatrix}.$$

$$A^{100} = (SBS^{-1})^{100} =$$
$$= (SBS^{-1})(SBS^{-1})\cdots(SBS^{-1}) =$$
$$= S B^{100} S^{-1} = SBS^{-1} = A.$$

Remark: A has eigenvectors $\vec{v}_1, \vec{v}_2$ $\quad \mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$

λ_1, λ_2

A

$$B = \begin{bmatrix} [A(\vec{v}_1)]_{\mathfrak{B}} & [A(\vec{v}_2)]_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\mathfrak{S} \underset{S}{\overset{S^{-1}}{\longleftrightarrow}} \mathfrak{B}$$

Recall: $\text{gemu}(\lambda) = n - \text{rank}(A - \lambda I_n)$.

Example: Find all the values a, b, c for which $A = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.

$\lambda = 1$, $\lambda = 0$

algebraic multiplicity 2. $\quad$ algebraic multiplicity 1 $\quad$ Recall: $\text{gemu}(\lambda) \leq \text{almu}(\lambda)$

$$\dim(E_0) = \text{gemu}(0) = 1$$

$$\dim(E_1) = \text{gemu}(1) = 3 - \text{rank}\begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix} = 3 - \text{rank}\begin{bmatrix} a & b \\ -1 & c \end{bmatrix} =$$

$$= \begin{cases} 3-2 & \text{if } ac+b \neq 0 \leftarrow \text{invertible} \\ 3-1 & \text{if } ac+b = 0 \leftarrow \text{not invertible} \end{cases}$$

Sum of gemu's is:

2 $\quad$ if $ac + b \neq 0$ $\longleftarrow$ No to eigenbasis.

3 $\quad$ if $ac + b = 0$ $\longleftarrow$ Yes to eigenbasis.

$$\begin{bmatrix} a & -b \end{bmatrix} \qquad \begin{bmatrix} a+ib & 0 \end{bmatrix}$$

<u>Remark:</u> 1. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is similar to $\begin{bmatrix} \text{[illegible]} & \text{[illegible]} \\ 0 & a-ib \end{bmatrix}$.

2. If A has eigenvalues $a \pm ib$ then A is similar to $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.