6. Symmetric matrices and quadratic forms.

We were interested in finding eigenbasis.

We say that a matrix is <u>diagonalizable</u> when it has an eigenbasis.

We are now interested in finding orthonormal eigenbasis.

We say that a matrix is <u>orthogonally diagonalizable</u> if it has an orthonormal eigenbasis.

$$A = SBS^{-1} = SBS^{T}$$

if $\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is orthonormal with $S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$, then S is orthogonal $S^{-1} = S^T$

<u>Theorem:</u> (Spectral Theorem) A matrix is orthogonally diagonalizable if and only if it is symmetric.

<u>Remark:</u> $\Rightarrow$) If A is $\underbrace{\text{orthogonally diagonalizable}}_{A = SDS^T,\ D \text{ diagonal}}$ then A is $\underbrace{\text{symmetric}}_{A = A^T}$.

$$A^T = (A)^T = (SDS^T)^T = (S^T)^T D^T S^T = SDS^T = A.$$

<u>Question:</u> Why does orthogonality preserve eigenspaces?

<u>Answer:</u> Having different eigenvalues implies orthogonality.

Theorem: Let A be a symmetric matrix, $\vec{v}_1$ and $\vec{v}_2$ eigenvectors with distinct eigenvalues λ_1 and λ_2. Then $\vec{v}_1 \cdot \vec{v}_2 = 0$, namely $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. In particular, they are linearly independent.

Recall: Eigenvectors with different eigenvalues are linearly independent.

Proof: $A \quad A = A^T$, $\vec{v}_1 \quad A\vec{v}_1 = \lambda_1 \vec{v}_1$, $\vec{v}_2 \quad A\vec{v}_2 = \lambda_2 \vec{v}_2$.

We want $\vec{v}_1 \cdot \vec{v}_2 = 0$.

$$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \cos(\theta)$$

$$\vec{v}_1 = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \qquad \vec{v}_1 \cdot \vec{v}_2 = a_1 b_1 + \cdots + a_n b_n$$

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{bmatrix} a_1 \cdots a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \vec{v}_1^T \vec{v}_2$$

$\vec{v}_1^T A \vec{v}_2$

(a) $\vec{v}_1^T (A \vec{v}_2) = \vec{v}_1^T (\lambda_2 \vec{v}_2) = \lambda_2 \vec{v}_1^T \vec{v}_2$

(b) $(\vec{v}_1^T A) \vec{v}_2 = (\vec{v}_1^T A^T) \vec{v}_2 = (A \vec{v}_1)^T \vec{v}_2 = (\lambda_1 \vec{v}_1)^T \vec{v}_2 = \lambda_1 \vec{v}_1^T \vec{v}_2$

So:

$$\lambda_2 \vec{v}_1^T \vec{v}_2 = \vec{v}_1^T A \vec{v}_2 = \lambda_1 \vec{v}_1^T \vec{v}_2 \rightsquigarrow \lambda_2 \vec{v}_1^T \vec{v}_2 - \lambda_1 \vec{v}_1^T \vec{v}_2 = 0$$

$$\rightsquigarrow \underbrace{(\lambda_2 - \lambda_1)}_{\text{distinct}} \vec{v}_1^T \vec{v}_2 = 0 \quad \rightsquigarrow \quad \vec{v}_1^T \vec{v}_2 = 0 \quad \text{so} \quad \vec{v}_1 \cdot \vec{v}_2 = 0. \qquad \square.$$

Method for computing orthonormal eigenbasis:

1. Find the eigenvectors and eigenspaces.

2. Find a basis of each eigenspace. Use Gram-Schmidt to find an orthonormal eigenbase for each eigenspace.

3. Concatenate these eigenbases.

__Example:__ Find an orthonormal eigenbasis for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\lambda = 2 \qquad \lambda = 0$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

1. They are orthogonal. (linear independent)

2. They have length one.

3. They are eigenvectors.

__Example:__ Find an orthonormal eigenbasis of $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$.

$$\lambda = 1 \qquad \lambda = 0$$

$$\underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \quad \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad \vec{u}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\vec{v}_2^{\perp}$

$$\vec{u}_2 = \frac{\quad}{\|\vec{v}_2^{\perp}\|} \quad , \quad \vec{v}_2^{\parallel} = \mathrm{proj}_{\mathrm{span}(\vec{u}_1)}(\vec{v}_2) = (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1 = \cdots = \frac{\quad}{2}\begin{bmatrix} \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$