A quadratic form is a function $q(x_1, \dots, x_n)$ from $\mathbb{R}^n$ to $\mathbb{R}$ that is a linear combination of products of $x_i x_j$ for $i,j \in \{1, \dots, n\}$. A quadratic form can be expressed as $q(\vec{x}) = \vec{x}^T A \vec{x}$ for A a unique symmetric matrix.

matrix associated to q

Example: Consider the function:

$$q(x_1, x_2, x_3) = \frac{2}{3}\left(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 - x_1x_3\right),$$

is this a quadratic form? Yes, because it is a linear combination of $x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3$. What is the symmetric matrix associated to q?

$$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \qquad A = A^T \qquad q(\vec{x}) = \vec{x}^T A \vec{x}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = q(x_1, x_2, x_3) = \frac{2}{3}\left(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 - x_1x_3\right)$$

$$d\,x_1x_2 + d\,x_2x_1 = 2d\,x_1x_2$$

$$A = (a_{ij})_{i,j}$$

a_{ii} the coefficient of x_i^2

$a_{ij} = a_{ji}$ half the coefficient of x_ix_j for $i \neq j$.

$$A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

Since $q(\vec{x}) = \vec{x}^T A \vec{x}$ has A a symmetric matrix associated to it, we can apply the Spectral Theorem to A. This gives $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ an orthonormal eigenbasis of A, with eigenvalues $\lambda_1, \dots, \lambda_n$. We can write $\vec{x} = \underset{\uparrow}{c_1} \vec{v}_1 + \dots + \underset{\uparrow}{c_n} \vec{v}_n$

and now: $S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ $\quad$ $\mathcal{B}$-coordinates of $\vec{x}$

$$q(\vec{x}) = \vec{x}^T A \vec{x} = (c_1 \vec{v}_1 + \dots + c_n \vec{v}_n)^T (c_1 \lambda_1 \vec{v}_1 + \dots + c_n \lambda_n \vec{v}_n) =$$

$$= \lambda_1 c_1^2 + \dots + \lambda_n c_n^2 .$$

<u>Example:</u> Consider the quadratic form given by $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$.

We have that $q(0,0,0) = 0$. Is $x_1 = x_2 = x_3 = 0$ a local/global minimum/maximum or neither?

It is a global minimum.

A has eigenvalues $1, 1, 0$, so if $\mathcal{B}$ is an orthonormal eigenbasis of A then:

$$q(\vec{x}) = c_1^2 + c_2^2 .$$

Since $q(\vec{x}) \geq 0$, $x_1 = x_2 = x_3 = 0$ is a global minimum.

As long as $c_1 = c_2 = 0$, it doesn't matter what c_3 is, we will have a global minimum.

Let $q(\vec{x}) = \vec{x}^T A \vec{x}$ be a quadratic form. We say that A is <u>positive definite</u>

if $q(\vec{x}) > 0$ when $\vec{x} \neq \vec{0}$. We say that A is positive semidefinite if $q(\vec{x}) \geq 0$ for all $\vec{x}$. We say that A is indefinite if q takes positive and negative values.

Question: Let A non-invertible. Why is A not positive definite?

Theorem: A is positive definite if and only if its eigenvalues are all positive.

Let A be symmetric, $n \times n$. Set $A^{(i)}$, $i = 1, \dots, n$, the $i \times i$ submatrix obtained from A by deleting all rows and columns past the i-th ones. We call $A^{(i)}$ the principal submatrices of A.

Theorem: A is positive definite if and only if the determinants of all its principal submatrices are positive.

Example: Consider $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, we have:

$$\det(A^{(1)}) = \det\left[\tfrac{2}{3}\right] = \tfrac{2}{3} > 0,$$

$$\det(A^{(2)}) = \det\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \tfrac{1}{3} > 0,$$

$$\det(A^{(3)}) = \det\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} = 0, \text{ which is not positive.}$$

So A is not positive definite.

Rmk: If A is not invertible then A is not positive definite.

If $\ker(A) \neq \{\vec{0}\}$ we have non-zero $\vec{v}$ such that $A\vec{v} = 0$.

$q(\vec{v}) = \vec{v}^T A \vec{v} = \vec{v}^T \vec{0} = \vec{0}$, so A is <u>not</u> positive definite.