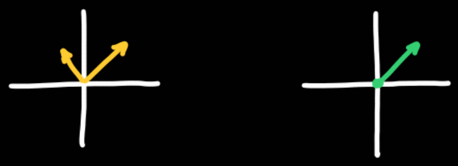
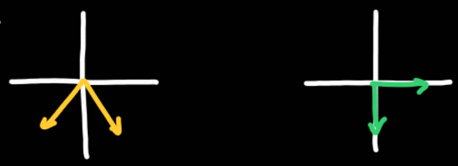
proj:



rotation:



Claim: Let A define a linear transformation. We can find an orthonormal basis of the source such that after applying A we have orthogonal vectors:

A $n \times m$ $\qquad \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$\vec{v}_1, \dots, \vec{v}_m \qquad A\vec{v}_1, \dots, A\vec{v}_m$

Example: Consider $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ that is orthonormal.

Hint: This is an application of the Spectral Theorem.

We need a symmetric matrix. $A^TA \rightsquigarrow$ symmetric.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda_1 = \dfrac{3+\sqrt{5}}{2}$ $\qquad \lambda_2 = \dfrac{3-\sqrt{5}}{2}$ $\qquad A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} \frac{\sqrt{5}}{2} - \frac{1}{2} \\ 1 \end{bmatrix}$ $\qquad \vec{v}_2 = \begin{bmatrix} -\frac{\sqrt{5}}{2} - \frac{1}{2} \\ 1 \end{bmatrix}$ $\qquad \ker(A^TA - \lambda I_2)$

normalize

$\dfrac{2}{1+\sqrt{5}} = \dfrac{\sqrt{5}-1}{2}$ $\qquad A^TA\vec{x} = \lambda\vec{x}$

$\vec{u}_1 = \begin{bmatrix} \frac{\sqrt{5-\sqrt{5}}}{\sqrt{10}} \\ \frac{\sqrt{2}}{\dots} \end{bmatrix}$ $\qquad \vec{u}_2 = \begin{bmatrix} -\frac{\sqrt{5+\sqrt{5}}}{\sqrt{10}} \\ \frac{\sqrt{2}}{\dots} \end{bmatrix}$ $\qquad \mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$.

$\mathbb{R}^2$

$\sqrt{5-\sqrt{5}}$] [$\sqrt{5+\sqrt{5}}$] $\mathbb{R}$

(b) Compute $L(\vec{u}_1)$, $L(\vec{u}_2)$, check that they are perpendicular.
$A\vec{u}_1$ $\quad$ $A\vec{u}_2$

$$A\vec{u}_1 = \begin{bmatrix} \sqrt{1+\frac{2}{\sqrt{5}}} \\ \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} \end{bmatrix} \qquad A\vec{u}_2 = \begin{bmatrix} -\sqrt{1-\frac{2}{\sqrt{5}}} \\ \frac{\sqrt{2}}{\sqrt{5+\sqrt{5}}} \end{bmatrix}$$

$$\frac{\sqrt{5-\sqrt{5}}}{\sqrt{10}} + \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} \cdot \frac{\sqrt{5-\sqrt{5}}}{\sqrt{5-\sqrt{5}}} = \cdots$$

$$\frac{1}{5-\sqrt{5}} \cdot \frac{5+\sqrt{5}}{5+\sqrt{5}}$$

$L(\vec{u}_1)$ is perpendicular to $L(\vec{u}_2)$.

$$\|L(\vec{u}_1)\|^2 = \frac{3}{2} + \frac{\sqrt{5}}{2} = \lambda_1 \quad , \quad \|L(\vec{u}_2)\|^2 = \lambda_2 .$$

The <u>singular values</u> $\sigma_1, \dots, \sigma_m$ of A are the square roots of the eigenvalues of $A^T A$, $\lambda_1, \dots, \lambda_m$.

<u>Theorem:</u> <u>Singular value decomposition:</u> Let A an $n\times m$ matrix. Then we can decompose it as:

$$\overset{n\times m}{A} = \underbrace{\overset{n\times n}{U}}_{\text{orthogonal matrix}} \underset{n\times m}{\Sigma} \underbrace{\overset{m\times m}{V^T}}_{\text{orthogonal matrix}}$$

Σ is a matrix with all zeroes except in the diagonal entries, where it has the singular values of A.

Method: A $n \times m$ $\quad r = \text{rank}(A)$

1. Find an orthonormal eigenbasis $\vec{v}_1, \dots, \vec{v}_m$ of $A^T A$.

$\mathbb{R}^m$

$\lambda_1, \dots, \lambda_m$

$\underbrace{\sigma_1, \dots,}_{r} \underbrace{\sigma_m}_{m-r}$

$\vec{u}_1$

notation in Example

$L(\vec{u}_1)$

$\mathbb{R}^n$

$A\vec{v}_1, \dots, A\vec{v}_m$

$\underbrace{\vec{u}_1, \dots, \vec{u}_r}_{r}$ $\qquad \vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i$

$\vec{u}_{r+1}, \dots, \vec{u}_r$

$$\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix} \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_m & - \end{bmatrix} = A.$$