To do:

1. Bases, notation, how to find them, linear transformation.

2. Drawing sketches for least-squares solutions.

3. Kernel and image of a matrix.

1. Bases, notation, how to find them, linear transformation.

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \longleftrightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{x} = c_1 \cdot \vec{v}_1 + \cdots + c_m \cdot \vec{v}_m = \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}}_{[\vec{x}]_{\mathfrak{B}}} =$$

$\vec{x}$ in $\mathbb{R}^n$

$$= S[\vec{x}]_{\mathfrak{B}}$$

$\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$

?

is this a basis of V $\qquad$ $\dim(V) = m$.

(a) $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$.

(b) $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.

Practice Final 7: $\quad x_1 + 2x_2 + x_3 = 0$, find $\mathfrak{B}$ with $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\circledast}$.

$\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$

$\mathbb{R}^3$

$\circledast$ $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2\vec{v}_1 - \vec{v}_2 \qquad \vec{v}_2 = 2\vec{v}_1 - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \qquad \vec{v}_2 = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix}$

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

use

choice

choice

$x_1 + 2x_2 + x_3 = 0$

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix} \right\}$$

$$2 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Practice Final 8: Find the matrix associated to the linear transformation:

$$L(\vec{x}) = \underbrace{\begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{in the basis } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}.$$

$$\begin{tikzcd} \mathbb{R}^2, \mathcal{E} \arrow[r, "A"] & \mathbb{R}^2, \mathcal{E} \\ \mathbb{R}^2, \mathcal{B} \arrow[r, "B"] \arrow[u, "S"] & \mathbb{R}^2, \mathcal{B} \arrow[u, "S"'] \end{tikzcd}$$

$$A = S B S^{-1}, \quad B = S^{-1} A S$$

$$A \neq S^{-1} B S \qquad B \neq S A S^{-1}$$

$$S = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \quad S^{-1} = \frac{1}{1} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} [A\vec{v}_1]_{\mathcal{B}} & [A\vec{v}_2]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & [L(\vec{v}_2)]_{\mathcal{B}} \end{bmatrix}$$

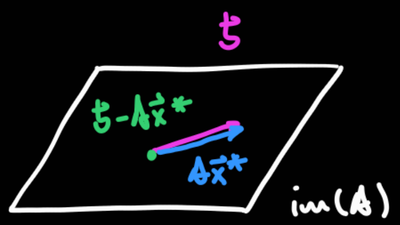
Recall: $L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation.

$$A = \begin{bmatrix} | & | \\ L(\vec{e}_1) & L(\vec{e}_2) \\ | & | \end{bmatrix}$$

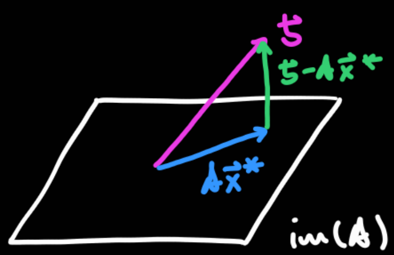
2. Drawing sketches:

Draw things for least-squares solutions: $A\vec{x} = \vec{b}$ $\quad \vec{x}^*$ $\quad \underbrace{A\vec{x}^*}_{\text{least-squares solution}} = \text{proj}_{\text{im}(A)}(\vec{b})$.

$A\vec{x}^*$, $\vec{b}$, $\text{im}(A)$, $\vec{b} - A\vec{x}^*$



if $\vec{b} \in \text{im}(A)$.



if $\vec{b} \notin \text{im}(A)$.

$$\vec{x} = \underbrace{\vec{x}^{\perp}}_{b - A\vec{x}^*} + \underbrace{\vec{x}^{\parallel}}_{A\vec{x}^*}$$

Practice Final 10: Find $\vec{x}^*$ for $A\vec{x} = \vec{b}$ with $A = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{\ker(A) \neq \{\vec{0}\}}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

If $\ker(A) = \{\vec{0}\}$ then:

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

Because we are solving the normal equation: $A^T A \vec{x} = A^T \vec{b}$.

A proof of why if $\ker(A) = \{\vec{0}\}$ then $A^T A$ is invertible can be found on the office hours notes/recordings.