Topics:

1. Basis: given a basis, compute the coordinate vectors, and vice versa.

2. Matrix of a linear transformation in a given basis.

3. Computing basis for the image and kernel of a matrix, and interpreting it geometrically.

1. Basis: given a basis, compute the coordinate vectors, and vice versa.

$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ basis of V subspace of $\mathbb{R}^n$

Notation from Jan. 26.:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}_{\mathcal{B}} = x_1 \cdot \vec{v}_1 + x_2 \cdot \vec{v}_2 + \cdots + x_m \cdot \vec{v}_m \qquad x_1, \dots, x_m \in \mathbb{R}$$

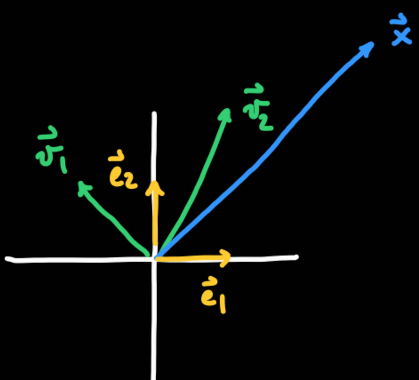
Notation from book:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \vec{x} = c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \cdots + c_m \cdot \vec{v}_m$$

$c_1, \dots, c_m \in \mathbb{R}$

Definition 3.4.1. page 149.

Example:



$$\mathcal{S} = \left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\} \qquad \mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

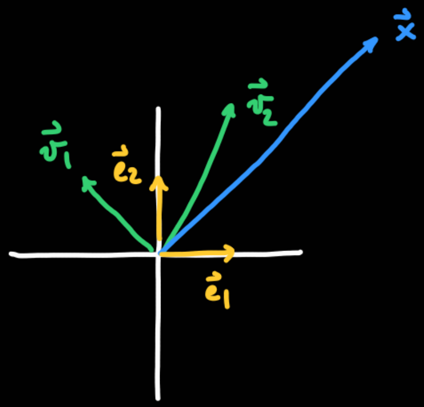
$S:\quad [\vec{x}]_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{\underbrace{S}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \vec{x} = c_1 \cdot \underset{\substack{\uparrow \\ S}}{\vec{e}_1} + c_2 \cdot \underset{\substack{\uparrow \\ S}}{\vec{e}_2}$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}_S = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad c_1 = 3 \quad c_2 = 3$$

$\mathcal{B}:\quad \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 3 \\ 3 \end{bmatrix} = d_1 \cdot \underset{\substack{\uparrow \\ \mathcal{B}}}{\vec{v}_1} + d_2 \cdot \underset{\substack{\uparrow \\ \mathcal{B}}}{\vec{v}_2} = d_1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + d_2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$d_1 = -1 \qquad d_2 = 2$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



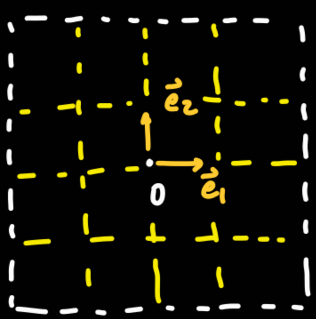
$$3 \cdot \vec{e}_1 + 3 \cdot \vec{e}_2 = -1 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2$$

S will be a matrix that intakes vectors in basis $\mathcal{B}$ and returns vectors in basis S.

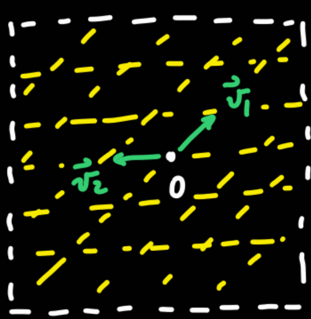
$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 3 \\ 3 \end{bmatrix} = -1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}}_{S} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$_S C_{\mathcal{B}}$

$\mathbb{R}^2$



$\mathbb{R}^2$



$$S = \begin{bmatrix} -1 & 1 \\ \end{bmatrix} = \begin{bmatrix} \vert & \vert \\ \vec{v}_1 & \vec{v}_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Example: $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ consider the vector $\vec{x} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$, write it in terms of the basis $\mathcal{B}$.

$$\begin{bmatrix} 5 \\ 7 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 5 \\ 7 \end{bmatrix} = c_1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\longleftrightarrow \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}}_{S \text{ change of basis matrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$S^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = S^{-1} S \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} 3 \\ -12 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Example: Let $\begin{bmatrix} 2 \\ 9 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, find $\mathcal{B} = \{ \vec{v}_1, \vec{v}_2 \}$.

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 9 \end{bmatrix} = S \begin{bmatrix} 1 \\ 5 \end{bmatrix} \qquad\qquad S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} = S \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

One way of solving this is writing out all four equations, and solving the system of linear equations.

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} a + 5b \\ c + 5d \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2a + 7b \\ 2c + 7d \end{bmatrix}$$

$$\begin{array}{cccc|c} a & b & c & d & \\ \hline 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 1 & 5 & 9 \\ 2 & 7 & 0 & 0 & 3 \\ 0 & 0 & 2 & 7 & 6 \end{array}$$

Another way: rewriting two equalities into one.

$$\begin{bmatrix}2\\9\end{bmatrix} = S\begin{bmatrix}1\\5\end{bmatrix}$$
$$\begin{bmatrix}3\\6\end{bmatrix} = S\begin{bmatrix}2\\7\end{bmatrix}$$
$$\rightsquigarrow \quad S\begin{bmatrix}1 & 2\\5 & 7\end{bmatrix} = \left[S\begin{bmatrix}1\\5\end{bmatrix} \; S\begin{bmatrix}2\\7\end{bmatrix}\right] = \begin{bmatrix}2 & 3\\9 & 6\end{bmatrix}$$

Theorem 2.3.2 page 78

$$S = S\begin{bmatrix}1 & 2\\5 & 7\end{bmatrix}\begin{bmatrix}1 & 2\\5 & 7\end{bmatrix}^{-1} = \begin{bmatrix}2 & 3\\9 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\5 & 7\end{bmatrix}^{-1} = \begin{bmatrix}2 & 3\\9 & 6\end{bmatrix}\begin{bmatrix}7 & -2\\-5 & 1\end{bmatrix}\frac{1}{-3} =$$
$$= \frac{-1}{3}\begin{bmatrix}-1 & -1\\33 & -12\end{bmatrix} = \begin{bmatrix}1/3 & 1/3\\-11 & 4\end{bmatrix}$$

so $\mathfrak{A} = \left\{\begin{bmatrix}1/3\\-11\end{bmatrix}, \begin{bmatrix}1/3\\4\end{bmatrix}\right\}$

<u>Recall:</u> A an $n\times n$ matrix:

$$A A^{-1} = I_n \quad , \quad A^{-1}A = I_n$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

<u>Example:</u> Let $\mathfrak{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\right\}$, $\mathfrak{R} = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$, find P such that $[\vec{x}]_{\mathfrak{R}} = P[\vec{x}]_{\mathfrak{B}}$.

P inputs coordinates in $\mathfrak{B}$ and outputs coordinates in $\mathfrak{R}$.

$$\underbrace{S_{\mathfrak{R}}}_{\text{matrix changing basis from } \mathfrak{R} \text{ to } \mathfrak{S}}[\vec{x}]_{\mathfrak{R}} = \vec{x} \longleftrightarrow [\vec{x}]_{\mathfrak{R}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} \qquad S_{\mathfrak{R}} = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$$

$$\underbrace{S_{\mathfrak{B}}}_{\text{matrix changing basis from } \mathfrak{B} \text{ to } \mathfrak{S}}[\vec{x}]_{\mathfrak{B}} = \vec{x} \longleftrightarrow [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix}d_1\\d_2\end{bmatrix} \qquad S_{\mathfrak{B}} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$$

So :

$$S_{\mathcal{R}}\,[\vec{x}]_{\mathcal{R}} = \vec{x} = S_{\mathcal{B}}\,[\vec{x}]_{\mathcal{B}} \rightsquigarrow [\vec{x}]_{\mathcal{R}} = \underbrace{S_{\mathcal{R}}^{-1} S_{\mathcal{B}}}_{P}\,[\vec{x}]_{\mathcal{B}}$$

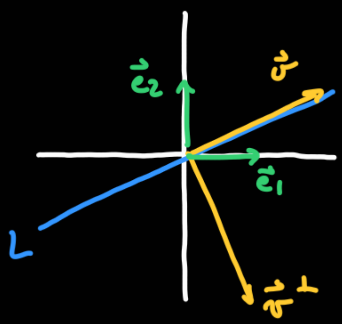
So :

$$P = S_{\mathcal{R}}^{-1} S_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} =$$

$$= \frac{-1}{2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$$

2. Matrix of a linear transformation in a given basis.

Example:



$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{v}^{\perp} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \mathcal{S} = \{\vec{e}_1, \vec{e}_2\},\ \mathcal{B} = \{\vec{v}, \vec{v}^{\perp}\}$$

Projection onto L :

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_1) = \left(\vec{e}_1 \cdot \frac{\vec{v}}{\|\vec{v}\|}\right) \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_2) = \left(\vec{e}_2 \cdot \frac{\vec{v}}{\|\vec{v}\|}\right) \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$$

$$T(\vec{v}) = \vec{v}$$

$$T(\vec{v}^{\perp}) = \vec{0}$$

$$B = \Big[[T(\vec v)]_{\mathfrak{B}} \;\; [T(\vec v^{\perp})]_{\mathfrak{B}} \Big] = \begin{bmatrix} \cdot & \cdot \\ 0 & 0 \end{bmatrix}$$

$$[T(\vec v)]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longleftrightarrow T(\vec v) = c_1 \cdot \vec v + c_2 \cdot \vec v^{\perp}$$

$$c_1 = 1 \quad c_2 = 0$$

$$[T(\vec v^{\perp})]_{\mathfrak{B}} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad \dots \qquad d_1 = 0 \quad d_2 = 0$$

Example: $\mathbb{R}^3$

$$\mathfrak{B} = \left\{ \underset{\vec v_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec v_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{\vec v_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\} \qquad V = \text{span}(\vec v_1, \vec v_3)$$

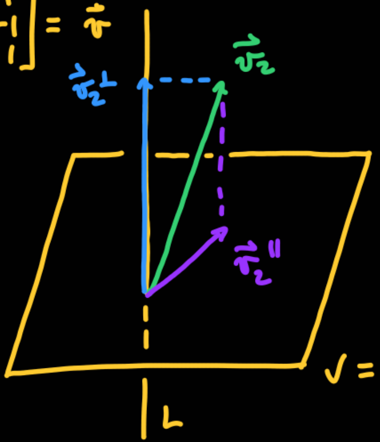
$T(\vec v_1) = \vec v_1$ $\qquad$ T projection onto V

$T(\vec v_3) = \vec v_3$ $\qquad$ $\vec v_2 = \frac{1}{3}\vec v_1 + \frac{2}{3}\vec v + \frac{1}{3}\vec v_3$

$$T(\vec v_2) = \vec v_2 - \text{proj}_L(\vec v_2) = \vec v_2 - \left(\vec v_2 \cdot \frac{\vec v}{\|\vec v\|}\right)\frac{\vec v}{\|\vec v\|} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{\sqrt3}\cdot\frac{1}{\sqrt3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \vec v$$



$\vec v_2 = \vec v_2^{\parallel} + \vec v_2^{\perp}$ so $\vec v_2^{\parallel} = \vec v_2 - \vec v_2^{\perp}$

$$V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

1. compute $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
2. $\vec v \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$

   $\vec v \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$

$$B = \Big[[T(\vec v_1)]_{\mathfrak{B}} \;\; [T(\vec v_2)]_{\mathfrak{B}} \;\; [T(\vec v_3)]_{\mathfrak{B}} \Big] = \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

$$[T(\vec v_2)]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = T(\vec v_2) = \underset{1/3 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{c_1} \cdot \vec v_1 + \underset{0 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{c_2} \cdot \vec v_2 + \underset{1/3 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{c_3} \cdot \vec v_3$$

$$[T(\vec v_2)]_{\mathfrak{B}} = \begin{bmatrix} 1/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

intakes $\mathcal{B}$, returns $\mathcal{S}$

To find S^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)\cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_2]{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}\cdot R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1 \end{array}\right] \xrightarrow[R_1 + R_3]{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1 \end{array}\right]$$

$$S^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 1 \\ -1/2 & 1/2 & 1 \end{bmatrix}$$

3. Geometric interpretation.

Algebraic: span(—), equation, $\{\cdots \mid \cdots\}$

Geometric: $\vec{0}$, 

point line plane 3D space