Problem 1.2.50:

$$x_2 = \tfrac{1}{2}(x_1 + x_3) \qquad\qquad -\tfrac{1}{2}x_1 + x_2 - \tfrac{1}{2}x_3 \qquad = 0$$

$$x_3 = \tfrac{1}{2}(x_2 + x_4) \quad \rightsquigarrow \quad -\tfrac{1}{2}x_2 + x_3 - \tfrac{1}{2}x_4 \qquad = 0$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$x_{n-1} = \tfrac{1}{2}(x_{n-2} + x_n) \qquad\qquad -\tfrac{1}{2}x_{n-2} + x_{n-1} - \tfrac{1}{2}x_n = 0$$

$$\begin{array}{l} k=2 \\ k=3 \\ k=4 \\ \\ \\ k=n-2 \\ k=n-1 \end{array}
\begin{bmatrix}
-\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & & & & & \\
0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & & & & & \\
0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & & & & & \\
\vdots & & & & & \ddots & & & & & \\
0 & & & & & & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\
0 & & & & & & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2}
\end{bmatrix}$$

$n-2$ rows

n columns

rank $= n-2$, we have two free variables.

We can choose $x_1 = t$ to be a free variable.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$x_2 = \tfrac{1}{2}(x_1 + x_3) = \tfrac{1}{2}\cdot t + \tfrac{1}{2}x_3$$

$x + y = 0 \rightsquigarrow y = -x$

$y + z = 0 \quad z = -y$

$(t, -t, t)$ x free

$(-t, t, -t)$ y free

We can choose $x_2 = s$ to be a free variable.

$k=2 \qquad x_3 = 2s - t.$

$k=3 \qquad x_3 = \tfrac{1}{2}(x_2 + x_4)$

$$2s - t = \tfrac{1}{2}(s + x_4) \qquad x_4 = 4s - s - 2t = 3s - 2t$$

$x_k = (k-1)\cdot s - (k-2)\cdot t$

$x_n = (n-1)\cdot s - (n-2)\cdot t$

<u>Claim:</u> $x_k = a\cdot s + b\cdot t$ with a, b integers depending only in k.

Final answer:

special case of this

$$(x_1, x_2, x_3, \dots, x_k, \dots, x_n) = (t, s, 2s-t, \dots, \underbrace{a\cdot s + b\cdot t}_{x_k}, \dots, \underbrace{\square}_{x_n}).$$

$x_k = a\cdot x_{k-1} + b\cdot x_{k+1}$ ← NOT valid answer

Problem 2.1.13.:

$$\vec{y} = T(\vec{x}) = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\vec{x} \quad \text{so} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What T inputs is $\vec{x}$, outputs $\vec{y}$. Inverting is doing the opposite: we input $\vec{y}$ and output $\vec{x}$.

Before: we know $\vec{x}$, we know A, we find $\vec{y}$.

After: we know $\vec{y}$, we know A, we find $\vec{x}$.

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\text{known}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{unknown}} \text{ translates into } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}, \text{ namely}$$

$$\begin{aligned} ax_1 + bx_2 &= y_1 \\ cx_1 + dx_2 &= y_2 \end{aligned}$$

To solve this, first assume $a \neq 0$. Then, we can divide the first equation by a:

$$x_1 + \frac{b}{a}x_2 = \frac{y_1}{a} \xrightarrow{E_2 - c\cdot E_1} x_1 + \frac{b}{a}x_2 = \frac{y_1}{a}$$

$cx_1 + dx_2 = y_2$ $\quad\quad$ 0 $\left(d - \frac{bc}{a}\right)x_2 = y_2 - \frac{c \cdot y_1}{a}$ ← *

We can simplify the bottom equation* if and only if $d - \frac{bc}{a} \neq 0$:

$$x_2 = \frac{y_2 - \frac{c \cdot y_1}{a}}{d - \frac{bc}{a}} = \frac{ay_2 - cy_1}{ad - bc}.$$

dividing by a

Note that $d - \frac{bc}{a} \neq 0$ if and only if $ad - bc \neq 0$.

multiply by a

If $a = 0$, the system is:

$$\begin{array}{r} bx_2 = y_1 \\ cx_1 + dx_2 = y_2 \end{array} \xrightarrow{\text{swap}} \begin{array}{r} cx_1 + dx_2 = y_2 \\ bx_2 = y_1 \end{array}$$

We can solve this if and only if $b \neq 0$ and $c \neq 0$. Then, since $a = 0$, the condition $ab - dc \neq 0$ is equivalent to $bc \neq 0$.

For part b), completely solve the system (knowing that $ad - bc \neq 0$).

$$x_1 = \frac{dy_1 - by_2}{ad - bc}.$$

$$x_2 = \frac{-cy_1 + ay_2}{ad - bc}.$$

Problem 2.1.49.:

row $\quad$ column

A is transition matrix if and only if $\sum_{i=1}^{n} a_{ij} = 1$ for all $j = 1, \dots, n$.

$\vec{x}$ is a distribution vector if and only if $\sum_{i=1}^{n} x_i = 1$.

Prove that $A\vec{x}$ is a distribution vector.

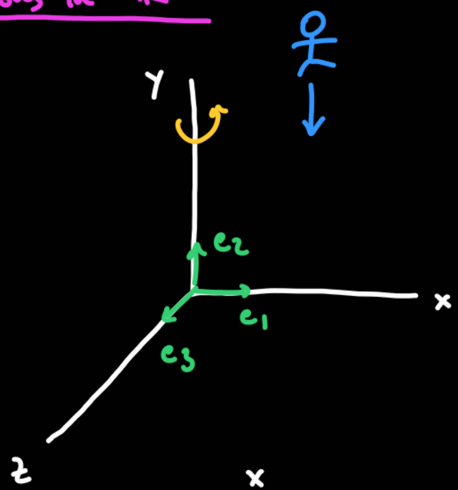
column $\quad$ ith entry

$A\vec{x} = \begin{bmatrix} \vdots \\ \sum_{j=1} a_{ij}x_j \\ \vdots \end{bmatrix}$, so the sum of the entries of $A\vec{x}$ is:

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}\,x_j =$$

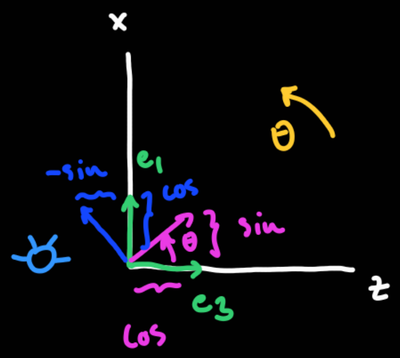
$$= \sum_{j=1}^{n}\underbrace{\left(\sum_{i=1}^{n} a_{ij}\right)}_{\substack{1 \\ A \text{ transition}}} x_j = \underbrace{\sum_{j=1}^{n} x_j}_{\vec{x} \text{ distribution}} = 1$$

Rotations in $\mathbb{R}^3$:



$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$T(e_1)$ $T(e_2)$ $T(e_3)$



$$\begin{array}{cc} \text{"x"} & \text{"y"} \end{array}$$
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Problem 2.1.14:

b) For which k do we have $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$ has integer entries.

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15}\begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix}$$

Since all entries are integers, then:

$$-a_{12} - a_{22} = \frac{3}{2k-15} - \frac{2}{2k-15} = \frac{1}{2k-15} \quad \text{is an integer.}$$

$$n = \frac{1}{2k-15} \quad \text{so} \quad 2k-15 = \frac{1}{n} \quad \text{so} \quad k = 7.5 - \frac{1}{2n}.$$

Moreover:

$$nk = 7.5n - \frac{1}{2}$$

$$nk = \frac{k}{2k-15}$$

so $\frac{k}{2k-15} = 7.5n - \frac{1}{2}$ is an integer.

So n must be odd.

If all the entries of the inverse are integers, then $k = 7.5 - \frac{1}{2n}$ for n some odd integer.

The converse is true: let $k = 7.5 - \frac{1}{2n}$ for n odd integer. Then:

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15}\begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{2(7.5-\frac{1}{2n})-15}\begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix} =$$

$$= n \cdot \begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} nk & -3n \\ -5n & 2n \end{bmatrix} = \begin{bmatrix} 7.5n - \frac{1}{2} & -3n \\ -5n & 2n \end{bmatrix}$$

which has all entries integers.

Better attempt to 1.2.50.:

$$x_k = \frac{1}{2}(x_{k-1} + x_{k+1}) \rightsquigarrow 2x_k = x_{k-1} + x_{k+1}$$

$$\rightsquigarrow x_k - x_{k-1} = x_{k+1} - x_k$$ for all consecutive pairs

$$x_1 - x_2 = x_2 - x_3$$

$$x_2 - x_3 = x_3 - x_4$$

$$\vdots$$

$$x_{n-1} - x_{n-2} = x_n - x_{n-1}$$

Arithmetic sequence:

$$t,\ t+r,\ t+2r,\ t+3r,\ \dots,\ t+(k-1)r,\ \dots,\ t+(n-1)r.$$

"General" rotations in $\mathbb{R}^3$: