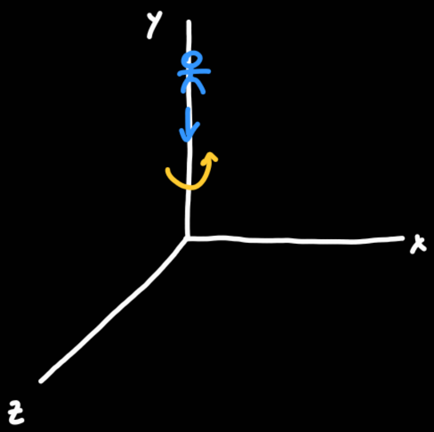
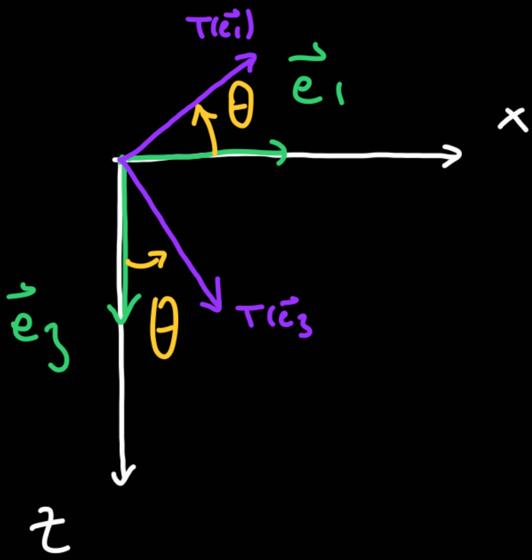
Midterm 1 Problem 4:



$$T(\vec{e}_1) = \begin{bmatrix} \cos(\theta) \\ 0 \\ -\sin(\theta) \end{bmatrix}$$

$$T(\vec{e}_2) = \vec{e}_2$$

$$T(\vec{e}_3) = \begin{bmatrix} \sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix}$$



Problem 3.3.28.:

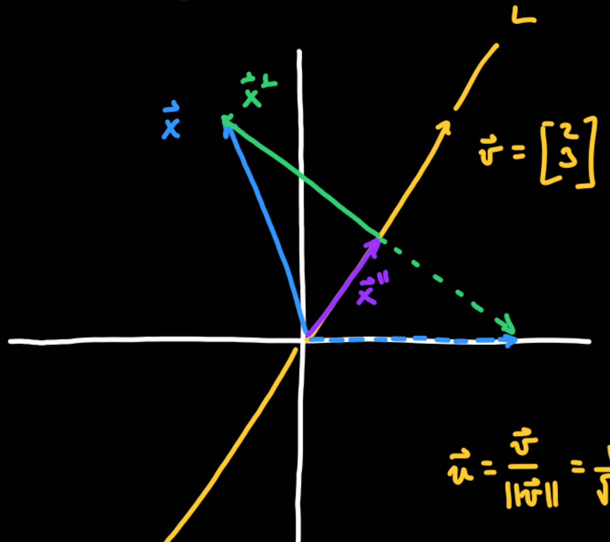
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 4 & k \end{bmatrix}}_{A} \xrightarrow[R_4 - 4R_3]{R_4 - 2R_1,\ R_4 - 3R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k-29 \end{bmatrix} \xrightarrow{\frac{R_4}{k-29}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\text{rref}(A) = I_4$

$k \neq 29$

Problem 3.4.38.: Matrix of reflection about $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.



$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$T(\vec{x}) = \vec{x} - 2\vec{x}^{\perp} = \vec{x}^{\parallel} - \vec{x}^{\perp}$$

$$T(\vec{e}_1) = \vec{e}_1 - 2 \cdot \frac{1}{13}\begin{bmatrix} 9 \\ -6 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

$$\vec{e}_1 = \vec{e}_1^{\parallel} + \vec{e}_1^{\perp}$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\vec{e}_1^{\parallel} = \text{proj}_L(\vec{e}_1) = (\vec{e}_1 \cdot \vec{u})\vec{u} =$$

$\vec{e}_1^{\,\parallel} = \mathrm{proj}_L(\vec{e}_1) = (\vec{e}_1 \cdot \vec{u}) \cdot \vec{u} =$

$= \frac{2}{\sqrt{13}} \cdot \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$

$\vec{e}_1^{\,\perp} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{13} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 9 \\ -6 \end{bmatrix}$

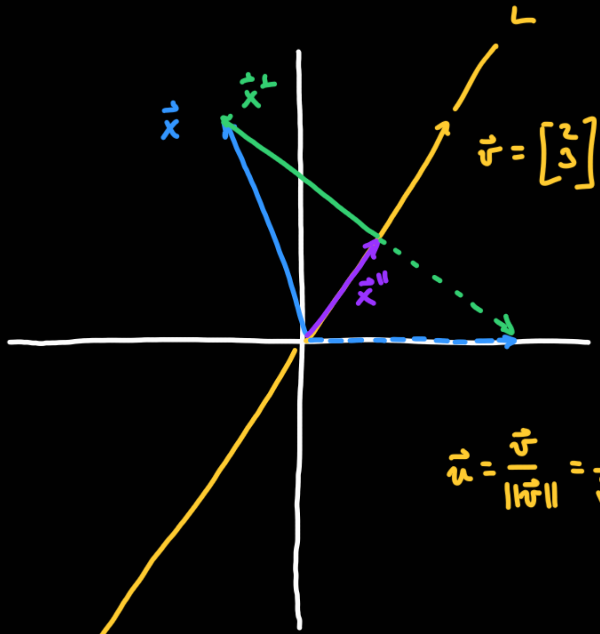
$T(\vec{e}_2) = \vec{e}_2 - 2 \cdot \frac{1}{13} \begin{bmatrix} -6 \\ 4 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 12 \\ 5 \end{bmatrix}$

$\vec{e}_2^{\,\parallel} = \frac{3}{\sqrt{13}} \cdot \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

$\vec{e}_2^{\,\perp} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{13} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -6 \\ 4 \end{bmatrix}$

$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -5 & -6 \\ 12 & 4 \end{bmatrix}$

Want: basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that the matrix of the reflection along $L = \mathrm{span}(\vec{v})$ is diagonal.



$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$

$T(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$

If we can find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that $\vec{u}_1$ is parallel to $\vec{v}$ and $\vec{u}_2$ is perpendicular to $\vec{v}$, then:

$T(\vec{u}_1) = \vec{u}_1$

$T(\vec{u}_2) = -\vec{u}_2$

$\mathfrak{B}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

in L in $L^{\perp}$

Choose $\mathfrak{B} = \{\vec{v}, \vec{w}\}$,

we find : $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$\begin{bmatrix} [T(\vec{v})]_{\mathfrak{B}} & [T(\vec{w})]_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{\mathfrak{B}} & \begin{bmatrix} -3 \\ 2 \end{bmatrix}_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$T(\vec{w}) = -\vec{w} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Problem 3.4.65.: Recall: A is similar to B if : $B = S^{-1} A S$ for S invertible matrix.

Prove that A an $n \times n$ matrix is similar to itself.

$A = S^{-1} A S$. Let $\underbrace{S = I_n}_{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}$, $S^{-1} = I_n$ now $I_n \cdot A \cdot I_n = A$.

Prove that if A is similar to B, then B is similar to A.

$B = S^{-1} A S$ for S some invertible matrix.

Want: $A = R^{-1} B R$ for some R invertible matrix.

Multiply by S on the LHS and by S^{-1} on the RHS :

$$S B S^{-1} = \underbrace{S\,(S^{-1}}_{I_n} A \underbrace{S)\, S^{-1}}_{I_n} = I_n A I_n = A$$

Rename $R = S^{-1}$, now $R^{-1} = (S^{-1})^{-1} = S$, so:

$$A = S B S^{-1} = R^{-1} B R \quad \text{with } R \text{ invertible matrix.}$$

So B is similar to A.

Problem 3.4.56.: Find $\mathfrak{B}$ such that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 4 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ means : $\vec{x} = c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 = \underbrace{\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}}_{S} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = S \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix}3\\4\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}2\\3\end{bmatrix} \longrightarrow \begin{bmatrix}3\\4\end{bmatrix} = S\begin{bmatrix}2\\3\end{bmatrix}$$

One method: $S = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, and the equalities above give 4 equations.

We can solve this system.

<u>Remember</u>: $BA = B\begin{bmatrix}| & & |\\ \vec{w}_1 & \cdots & \vec{w}_m\\ | & & |\end{bmatrix} = \begin{bmatrix}| & & |\\ B\vec{w}_1 & \cdots & B\vec{w}_m\\ | & & |\end{bmatrix}$

$$S\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$
$$S\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} \longrightarrow S\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix} = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix} \quad \text{so} \quad S = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}^{-1}$$