We know $(\mathrm{im}(A))^{\perp} = \ker(A^T)$ is always true.

We want a matrix A with $(\mathrm{im}(A))^{\perp} \neq \ker(A)$. So since $(\mathrm{im}(A))^{\perp} = \ker(A^T)$,

we want $\ker(A^T) = (\mathrm{im}(A))^{\perp} \neq \ker(A)$. Now:

$$A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$A^T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

For $\ker(A)$:

$A\vec{x} = \vec{0}$, so $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \end{array}\right]$ so $y = t$, $z = s$ are free and

$$\vec{x} = \begin{bmatrix} -2t-3s \\ t \\ s \end{bmatrix}$$ so $\ker(A) = \mathrm{span}\left(\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)$ has dimension 2.

For $\ker(A^T)$:

$A^T\vec{x} = \vec{0}$, so $\left[\begin{array}{c|c} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{array}\right]$ which reduces to $\left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$ so $x = 0$

so $\ker(A^T) = \{\vec{0}\}$.

If A is $n \times n$ then: (do we have $\ker(A) = \ker(A^T)$).

$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = n$ $\quad \dim(\mathrm{im}(A)) = \mathrm{rank}(A)$

$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = n$, $\dim(\mathrm{im}(A)) = \mathrm{rank}(A)$

$\dim(\mathrm{im}(A^T)) + \dim(\ker(A^T)) = n$, $\dim(\mathrm{im}(A^T)) = \mathrm{rank}(A^T)$

but $\mathrm{rank}(A) = \mathrm{rank}(A^T)$ so $\dim(\mathrm{im}(A)) = \dim(\mathrm{im}(A^T))$ so:

$\dim(\ker(A)) = \dim(\ker(A^T))$. No!. See the end remark.

Problem 5.4.3.: Let $\vec{v}_1, \dots, \vec{v}_p$ be a basis of V, $\vec{w}_1, \dots, \vec{w}_q$ a basis of $V^\perp$. Is $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ a basis of $\mathbb{R}^n$?

$\mathbb{R}^n$ is separated into V and $V^\perp$, so we think yes.

To prove it, we need $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ to be linearly independent and to span $(\mathbb{R}^n)$.

1. Suppose we have a sum:

$$c_1\vec{v}_1 + \dots + c_p\vec{v}_p + d_1\vec{w}_1 + \dots + d_q\vec{w}_q = 0$$

So:

$$\underbrace{c_1\vec{v}_1 + \dots + c_p\vec{v}_p}_{\text{in } V} = \underbrace{-(d_1\vec{w}_1 + \dots + d_q\vec{w}_q)}_{\text{in } V^\perp}$$

So:

$c_1\vec{v}_1 + \dots + c_p\vec{v}_p$ is in V and $V^\perp$

$-(d_1\vec{w}_1 + \dots + d_q\vec{w}_q)$ is in V and $V^\perp$

But there is only one vector in V and $V^\perp$, the vector $\vec{0}$:

$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = 0$ so $c_1 = \cdots = c_p = 0$ since $\vec{v}_1, \ldots, \vec{v}_p$ are linearly independent

$-(d_1\vec{w}_1 + \cdots + d_q\vec{w}_q) = 0$ so $d_1\vec{w}_1 + \cdots + d_q\vec{w}_q = 0$ so $d_1 = \cdots = d_q = 0$

since $\vec{w}_1, \ldots, \vec{w}_q$ are linearly independent.

Thus any sum $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p + d_1\vec{w}_1 + \cdots + d_q\vec{w}_q = 0$ always has all coefficients zero, so we cannot have linear dependence.

2. Let $\vec{x}$ be in $\mathbb{R}^n$, decompose $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, where $\vec{x}^{\parallel}$ is in V and $\vec{x}^{\perp}$ is in $V^{\perp}$, then $\vec{x}^{\parallel}$ is in $\text{span}(\vec{v}_1, \ldots, \vec{v}_p)$ and $\vec{x}^{\perp}$ is in $\text{span}(\vec{w}_1, \ldots, \vec{w}_q)$. Thus $\vec{x}$ is in $\text{span}(\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q)$. By definition:

$\mathbb{R}^n = \text{span}(\vec{v}_1, \ldots, \vec{v}_p, \vec{w}_1, \ldots, \vec{w}_q)$.

Problem 5.3.32.:

a) Let A be $n \times m$ with $A^TA = I_m$, do we have $AA^T = I_n$?

Thought process: Suppose that $m < n$, so
$$A : \begin{tikzcd} \mathbb{R}^m \arrow[r] & \mathbb{R}^n \arrow[l, bend left, "A^T"] \end{tikzcd}$$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

A will be a 3×2 matrix, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Now: $A^TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which is not I_3.

b) Let A be $n \times n$ with $A^T A = I_n$, do we have $AA^T = I_n$?

If $A^T A = I_n$ then A is orthogonal so $A^{-1} = A^T$ so $AA^T = AA^{-1} = I_n$.

<u>Remark:</u> Let A be $n \times n$ with $BA = I_n$, do we have $AB = I_n$?

<u>Yes</u>!

Here A will have $\ker(A) = \{\vec{0}\}$, so it will be invertible, and $A^{-1} = B$.

1. $\ker(A) = \{\vec{0}\}$:

Let $\vec{x} \in \ker(A)$, so $A\vec{x} = 0$. Then:

$$\vec{x} = I_n \vec{x} = (BA)\vec{x} = B(A\vec{x}) = B\vec{0} = \vec{0} \quad \text{so} \quad \vec{x} = \vec{0}.$$

2. $A^{-1} = B$:

Since $BA = I_n$ so multiplying by A^{-1} on the right:

$$BAA^{-1} = I_n A^{-1} \quad \text{so} \quad B = A^{-1}!$$

<u>Proof of Cramer's rule:</u> A invertible, $A\vec{x} = \vec{b}$, the solutions are $\vec{x} = \begin{bmatrix} \vdots \\ \frac{\det(A_{\vec{b},i})}{\det(A)} \\ \vdots \end{bmatrix}$.

We have to prove $x_i = \frac{\det(A_{\vec{b},i})}{\det(A)}$. Equivalently, we want to prove:

$$\det(A)\, x_i = \det(A_{\vec{b},i}) \quad \text{when} \quad A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

$$A_{\vec{b},i} = \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{b} & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & A\vec{x} & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix} =$$

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{v}_1 \cdot x_1 + \cdots + \vec{v}_n \cdot x_n$$

$$\begin{bmatrix} | & & | \end{bmatrix}\begin{bmatrix} x_n \end{bmatrix}$$

$$= \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{v}_1 \cdot x_1 + \cdots + \vec{v}_n \cdot x_n & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix}$$ so when taking determinants

$$\det(A_{\vec{b},i}) = \det \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{v}_1 \cdot x_1 + \cdots + \vec{v}_n \cdot x_n & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix} =$$

$$= \det \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & x_1 \vec{v}_1 & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix} + \cdots +$$

$$\det \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & x_i \vec{v}_i & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix} + \cdots + \det \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & x_n \vec{v}_n & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix} =$$

$$= x_1 \underbrace{\det \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{v}_1 & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix}}_{0} + \cdots +$$

$$x_i \det \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{v}_i & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix} + \cdots + x_n \underbrace{\det \begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{v}_n & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix}}_{0} =$$

So:

$$\det(A_{\vec{b},i}) = x_i \cdot \det \underbrace{\begin{bmatrix} | & & | & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{v}_i & \vec{v}_{i+1} & \cdots & \vec{v}_n \\ | & & | & | & | & & | \end{bmatrix}}_{A} = x_i \cdot \det(A)$$

Problem 5.4.25.:

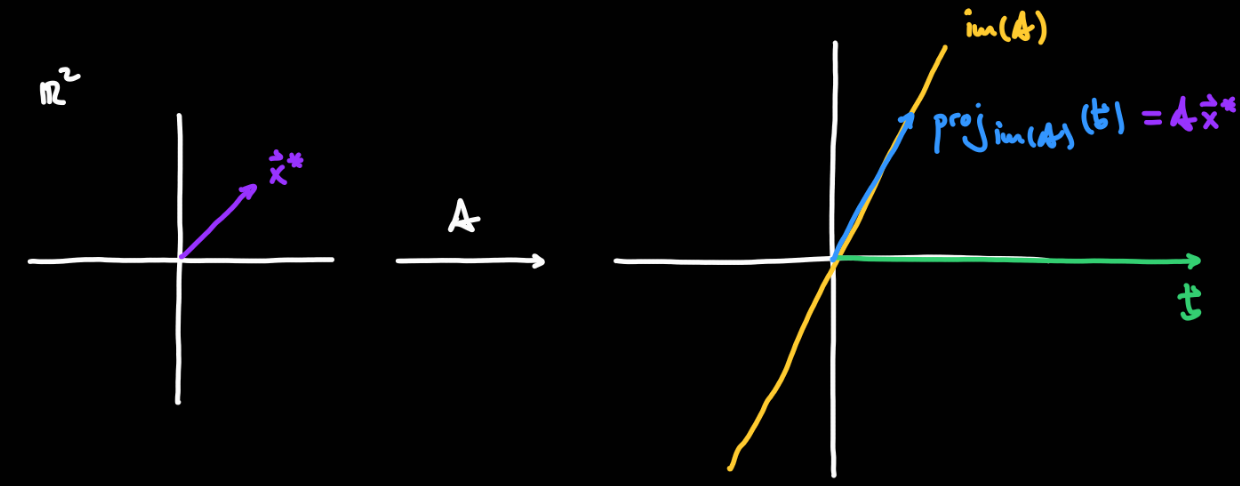
Since A does not have kernel $\vec{0}$, we project $\vec{b}$ onto $\operatorname{im}(A)$ and we solve for

$\vec{x}^*$ in the system $A\vec{x}^* = \operatorname{proj}_{\operatorname{im}(A)}(\vec{b})$. $\quad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

$$\vec{u} = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

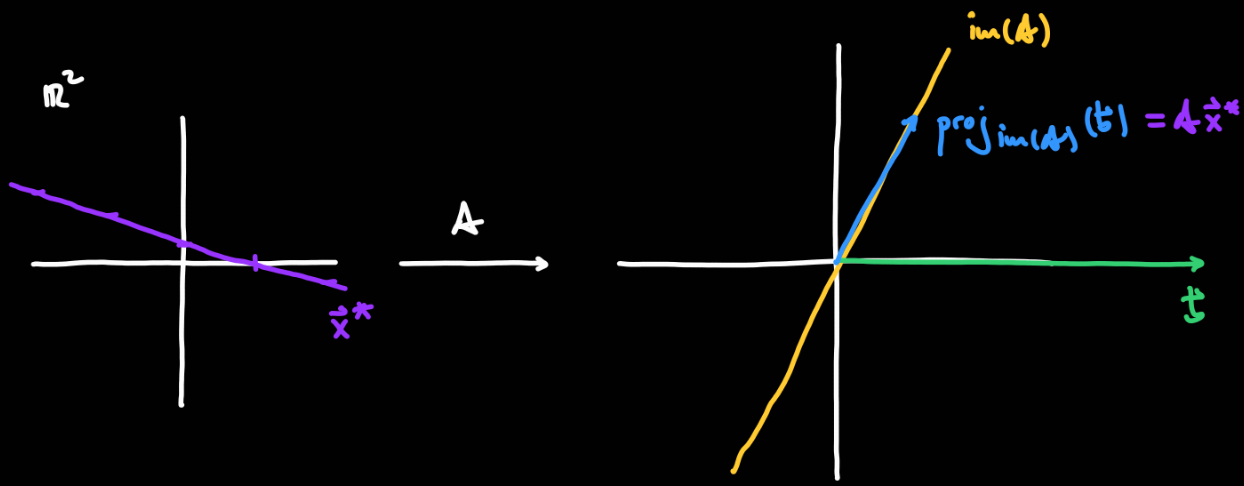
$$\text{proj}_{\text{im}(A)}(\vec{b}) = (\vec{b} \cdot \vec{u})\,\vec{u} = \frac{1}{5} \cdot 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



We want $\vec{x}^*$ in $\mathbb{R}^2$ such that $A\vec{x}^* = \vec{b}$.

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 6 & 2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad x + 3y = 1. \quad y = \frac{-x}{3} + \frac{1}{3}$$

Set $y = t$ free, then $\vec{x}^* = \begin{bmatrix} 1-3t \\ t \end{bmatrix}$.



<u>Remark:</u> In general, if A is $n \times n$, then $\ker(A) \neq \ker(A^T)$.

Let $n = 3$ and $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Now:

$$\ker(A) = \left\{ \begin{bmatrix} -2t-3s \\ t \\ s \end{bmatrix} \text{ for } t, s \text{ in } \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right),$$

$$\ker(A^T) = \left\{ \begin{bmatrix} 0 \\ t \\ s \end{bmatrix} \text{ for } t, s \text{ in } \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right).$$