$$A = U \Sigma \underbrace{V^T}_{\text{orthonormal eigenbasis of } A^TA}$$

$n \times m$

$V^{-1} = V^T$ and $U^{-1} = U^T$

$$\begin{tikzcd}
\mathbb{R}^m, \mathfrak{S} \arrow[r, "A"] \arrow[d, "V^T"'] & \mathbb{R}^n, \mathfrak{S} \\
\mathbb{R}^m, \mathfrak{B} \arrow[r, "\Sigma"'] & \mathbb{R}^n, \mathfrak{A} \arrow[u, "U"']
\end{tikzcd}$$

$$\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$$

$A\vec{v}_1$ has length $\|A\vec{v}_1\|^2 = (A\vec{v}_1)\cdot(A\vec{v}_1) = (A\vec{v}_1)^T(A\vec{v}_1) =$

$= \vec{v}_1^T A^T A \vec{v}_1 = \vec{v}_1^T(\lambda_1 \vec{v}_1) = \lambda_1 \vec{v}_1^T \vec{v}_1 =$

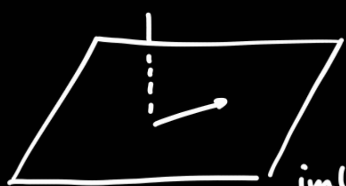
$= \lambda_1 (\vec{v}_1 \cdot \vec{v}_1) = \lambda_1$

$$\overset{n\times n}{A} = S \underset{n\times n}{\overset{n\times n}{B}} \underset{n\times n}{S^{-1}}$$

<u>Singular value decomposition.</u>

## <u>Practice Final 10.:</u>

$\vec{b}$, $A\vec{x}^*$, $\vec{b} - A\vec{x}^*$, $\mathrm{im}(A)$



$\mathrm{im}(A)$

$\vec{x}^*$ can be a whole subspace

$$\vec{x}^* = \begin{bmatrix} t - \frac{7}{6} \\ 1 - 2t \\ t \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A\vec{x}^* = \Big[\, t - \tfrac{7}{6} + 2 - 4t + 3t \qquad -\frac{7}{6} + 2 = \frac{12}{6} - \frac{7}{6} = \frac{5}{6}$$

## Practice Final 7.:

$x_1 + 2x_2 + x_3 = 0$ , $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ $\quad \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$

$$\underbrace{\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2 \cdot \vec{v}_1 - \vec{v}_2}$$

$$\vec{v}_2 = 2\vec{v}_1 - \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\vec{x}}$$

$\vec{v}_1$ should <u>not</u> be parallel to $\vec{x}$.

To find $\vec{v}_1$ we only need to choose two components: the third will be determined by the fact that $\vec{v}_1$ is in $x_1 + 2x_2 + x_3 = 0$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{array}{l} \leftarrow \text{chosen by us} \\ \leftarrow \text{chosen by us} \\ \leftarrow \text{given by } x_1 + 2x_2 + x_3 = 0 \end{array}$$

$$\vec{v}_2 = 2 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

$$S = [\vec{v}_1 \;\; \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -3 \end{bmatrix} \qquad \underbrace{S \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \begin{bmatrix} 2 - 1 \\ -1 \\ -2 + 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\vec{x}}$$

$$\boxed{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \quad \rightsquigarrow \quad S[\vec{x}]_{\mathcal{B}} = S \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \vec{x}}$$

$$\mathbb{R}^n, \mathcal{E} \xleftarrow{\;S\;} \mathbb{R}^m, \mathcal{B}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad\qquad \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

$$\mathbb{R}^3, \mathcal{E} \xleftarrow{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -3 \end{bmatrix}} \mathbb{R}^2, \mathcal{B}$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

## Singular value decomposition:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^T A: \quad \underbrace{\vec{v}_1 = \begin{bmatrix} \sqrt{5-\sqrt{5}}/\sqrt{10} \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -\sqrt{5+\sqrt{5}}/\sqrt{10} \\ \sqrt{2}/\sqrt{5+\sqrt{5}} \end{bmatrix}}_{\checkmark}$$

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2} \qquad\qquad \lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$$

$$\sigma_1 = \frac{\sqrt{5}}{2} + \frac{1}{2} \qquad\qquad \sigma_2 = \frac{\sqrt{5}}{2} - \frac{1}{2}$$

$$\frac{1}{2} + \frac{\sqrt{5}}{2} \overset{?}{=} \sqrt{\frac{3}{2} + \frac{\sqrt{5}}{2}} = \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{\sqrt{3+\sqrt{5}}}{\sqrt{2}}$$

$$\frac{1+\sqrt{5}}{2} \overset{?}{=} \frac{\sqrt{3+\sqrt{5}}}{\sqrt{2}} \qquad\qquad x = \sqrt{\frac{3+\sqrt{5}}{2}}$$

$$\frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} \qquad\qquad x^2 = \frac{3+\sqrt{5}}{2}$$

$$6 + 2\sqrt{5} = 2(3+\sqrt{5}) \ \checkmark \qquad 4x^2 = 6 + 2\sqrt{5} = (1+\sqrt{5})^2$$

$$x^2 = \frac{(1+\sqrt{5})^2}{4} = \left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$A\vec{v}_1 = \begin{bmatrix} \sqrt{1+\frac{2}{\sqrt{5}}} \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix} \qquad A\vec{v}_2 = \begin{bmatrix} -\sqrt{1-\frac{2}{\sqrt{5}}} \\ \sqrt{2}/\sqrt{5+\sqrt{5}} \end{bmatrix}$$

$$\underbrace{\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \begin{bmatrix} \sqrt{5+\sqrt{5}}/\sqrt{10} \\ \sqrt{5-\sqrt{5}}/\sqrt{10} \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \begin{bmatrix} -\sqrt{5-\sqrt{5}}/\sqrt{10} \\ \sqrt{5+\sqrt{5}}/\sqrt{10} \end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix} \sqrt{5+\sqrt{5}}/\sqrt{10} & -\sqrt{5-\sqrt{5}}/\sqrt{10} \\ \sqrt{5-\sqrt{5}}/\sqrt{10} & \sqrt{5+\sqrt{5}}/\sqrt{10} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \frac{1}{2}+\frac{\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}}{2}-\frac{1}{2} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \sqrt{5-\sqrt{5}}/\sqrt{10} & \sqrt{2}/\sqrt{5-\sqrt{5}} \\ -\sqrt{5+\sqrt{5}}/\sqrt{10} & \sqrt{2}/\sqrt{5+\sqrt{5}} \end{bmatrix}}_{V^T} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^T \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & b^2+a^2 \end{bmatrix}$$

$$\begin{matrix} a & b \\ -b & a \end{matrix}$$

$A, \vec{v} \qquad A\vec{v} = \lambda\vec{v} \qquad A^T = A$

$$\vec{v}^T A \vec{v} = \vec{v}^T \lambda \vec{v} = \lambda \cdot \|\vec{v}\|^2$$

$$\shortparallel$$

$$(A^T\vec{v})^T\vec{v} = (\lambda\vec{v})^T\vec{v} = \lambda \cdot \vec{v}^T\vec{v}$$

$\lambda$ eigenvalue of $A^TA$, eigenvector $\vec{v}$, then:

$$\|A\vec{v}\|^2 = (A\vec{v})\cdot(A\vec{v}) = (A\vec{v})^T(A\vec{v}) = \vec{v}^T A^T A\vec{v} = \lambda\vec{v}^T\vec{v} = \lambda \cdot \|\vec{v}\|^2$$