Math 31B
Integration and Infinite Series

Midterm 2

Instructions: You have 24 hours to complete this exam. There are 6 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. This document is the template where you need to provide your answers. Please print or download this document, complete it in the space provided, show your work in the space provided, clearly box your final answer, and upload a pdf version of this document with your solutions. Do **NOT** upload a different document, and do **NOT** upload loose paper sheets. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. **Failure to comply with any of these instructions may have repercussions in your final grade.**

Name: ______________________________

ID number: ______________________________

Section: ______________________________

Question	Points	Score
1	15	
2	17	
3	17	
4	17	
5	17	
6	17	
Total:	100	

Problem 1. *15pts.*

Determine whether the following statements are true or false. If the statement is true, write T in the box provided under the statement. If the statement is false, write F in the box provided under the statement. Do not write "true" or "false".

(a) **F** Let S be the solid obtained by rotating the region below a curve $f(x)$. If S has finite volume then S has finite surface area.

(b) **F** Given polynomials $p(x)$ and $q(x)$, then we can always find the partial fraction decomposition of $\frac{p(x)}{q(x)}$.

(c) **T** There are improper integrals that do not converge.

(d) **F** Let $f(x)$ be any function. The nth Taylor polynomial of $f(x)$ coincides with $f(x)$ for n large enough.

(e) **F** The Comparison Test can always be used to determine the convergence of $\sum_{n=1}^{\infty} a_n$ if we know the convergence of $\sum_{n=1}^{\infty} b_n$.

Problem 2. *17pts.*

Find the integral of $f(x) = \frac{3x^2-4x+5}{x^3-x^2+x-1}$ between 2 and 3.

Solution: We have the partial fraction decomposition:

$$\frac{3x^2-4x+5}{x^3-x^2+x-1} = \frac{2}{x-1} + \frac{x-3}{x^2+1}$$

and thus

$$\begin{aligned}
\int_2^3 \frac{3x^2-4x+5}{x^3-x^2+x-1}dx &= \int_2^3 \frac{2dx}{x-1} + \int_2^3 \frac{xdx}{x^2+1}dx - \int_2^3 \frac{3dx}{x^2+1} \\
&= 2\ln|x-1|\Big|_2^3 + \frac{1}{2}\ln|x^2+1|\Big|_2^3 - 3\arctan(x)\Big|_2^3 \\
&= \frac{1}{2}\left(\ln(32) + 6\arctan(2) - 6\arctan(3)\right).
\end{aligned}$$

Problem 3. *17pts.*

Determine whether $\int_0^\pi \sec(\theta)d\theta$ converges and, if so, evaluate it.

Solution: Note that $\sec(\theta)$ has a vertical asymptote at $\theta = \pi/2$. We then separate the integral as:

$$\int_0^\pi \sec(\theta)d\theta = \int_0^{\pi/2} \sec(\theta)d\theta + \int_{\pi/2}^\pi \sec(\theta)d\theta.$$

Now:

$$\begin{aligned}
\int_0^{\pi/2} \sec(\theta)d\theta &= \lim_{N\to\pi/2} \int_0^N \sec(\theta)d\theta \\
&= \lim_{N\to\pi/2} \ln|\sec(\theta) + \tan(\theta)|\Big|_0^N \\
&= \lim_{N\to\pi/2} \ln|\sec(N) + \tan(N)| \\
&= \infty.
\end{aligned}$$

Since one of the integrals in the expansion of $\int_0^\pi \sec(\theta)d\theta$ is not convergent, then by definition the whole integral also does not converge.

Problem 4. *17pts.*

Compute the surface area of revolution about the x-axis of $f(x) = (4 - x^{2/3})^{3/2}$ in $[0, 8]$.

Solution: We have:

$$f'(x) = \frac{-1}{x^{1/3}}(4 - x^{2/3})^{1/2}$$

so

$$1 + (f'(x))^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}.$$

Hence the surface area of revolution is

$$S = 2\pi \int_0^8 (4 - x^{2/3})^{3/2} \frac{2dx}{x^{1/3}} = 2\pi \int_4^0 (-3)u^{3/2} du = \frac{384}{5}\pi.$$

Problem 5. *17pts.*

(a) Find $T_5(x)$ for $f(x) = \cos(x)$ and evaluate it at $x = 0$.

(b) Find the maximum possible size of the error between $f(x) = \cos(x)$ and $T_5(x)$ around $a = 0$ when evaluated at $x = 0.25$.

Solution:

(a) We have the expansion around $x = 0$:

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and thus

$$T_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Hence $T(0) = 1$.

(b) We have $f^{(6)}(x) = -\cos(x)$ so $|f^{(6)}(x)| \leq 1$ for all real numbers x, meaning that we can choose $K = 1$ in the formula to bound the error. Then:

$$|\cos(0.5) - T_5(0.5)| = |f(x) - T_5(x)| \leq \frac{Kx^6}{6!} = \frac{\left(\frac{1}{4}\right)^6}{6!} = \frac{1}{2^{12} \cdot 6!} = \frac{1}{2949120}.$$

Problem 6. *17pts.*

(a) Compute the limit of the sequence with general term $a_n = (2^n + 3^n)^{1/n}$.

(b) Write $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ as a telescopic series and find its sum.

Solution:

(a) Note that $2^n + 3^n \geq 3^n$ so:

$$(2^n + 3^n)^{1/n} \geq (3^n)^{1/n} = 3.$$

Note that $2^n + 3^n \leq 3^n + 3^n = 2 \cdot 3^n$ so:

$$(2^n + 3^n)^{1/n} \leq (2 \cdot 3^n)^{1/n} = 2^{1/n} \cdot 3.$$

Since

$$\lim_{n\to\infty} 3 = 3 \quad \text{and} \quad \lim_{n\to\infty} 2^{1/n} \cdot 3 = 3$$

then by the Squeeze Theorem for sequences we have

$$\lim_{n\to\infty} (2^n + 3^n)^{1/n} = 3.$$

(b) We have:

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

and thus

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right) = \frac{1}{3}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right) = \frac{1}{3} \cdot \frac{11}{6} = \frac{11}{18}.$$