# Math 31B
# Integration and Infinite Series

## Practice Midterm 2

**Instructions:** You have 24 hours to complete this exam. There are 6 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. This document is the template where you need to provide your answers. Please print or download this document, complete it in the space provided, show your work in the space provided, clearly box your final answer, and upload a pdf version of this document with your solutions. Do **NOT** upload a different document, and do **NOT** upload loose paper sheets. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. **Failure to comply with any of these instructions may have repercussions in your final grade.**

Name: ______________________________

ID number: ______________________________

Section: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 17 | |
| 3 | 17 | |
| 4 | 17 | |
| 5 | 17 | |
| 6 | 17 | |
| Total: | 100 | |

**Problem 1**. *15pts.*

Determine whether the following statements are true or false. If the statement is true, write T in the box provided under the statement. If the statement is false, write F in the box provided under the statement. Do not write "true" or "false".

(a) **F** Let $S$ be the solid obtained by rotating the region below a curve $f(x)$. The volume of $S$ is always smaller than the surface area of $S$.

(b) **T** Given non-zero polynomials $p(x)$ and $q(x)$, then we can always compute $\int \frac{p(x)}{q(x)} dx$.

(c) **T** To compute improper integrals we can use limit(s).

(d) **T** Let $f(x)$ be any function. The $n$th Taylor polynomial of $f(x)$ is an approximation of $f(x)$ using the first $n$ derivatives of $f(x)$.

(e) **T** If we have a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ does not converge.

**Problem 2**. *17pts.*

Find the integral of $f(x) = \frac{10}{x^4-2x^3+10x^2-18x+9}$.

**Solution:** We have the partial fraction decomposition:

$$\frac{10}{x^4-2x^3+10x^2-18x+9} = \frac{\frac{-1}{5}}{x-1} + \frac{1}{(x-1)^2} + \frac{\frac{1}{5}x-\frac{4}{5}}{x^2+9}$$

and thus

$$\int \frac{10dx}{x^4-2x^3+10x^2-18x+9} = \int \frac{-dx}{5(x-1)} + \int \frac{dx}{(x-1)^2} + \int \frac{(x-4)dx}{5(x^2+9)}$$
$$= \frac{-1}{5}\ln|x-1| - \frac{1}{x-1} + \frac{1}{10}\ln|x^2+9| - \frac{4}{15}\arctan\left(\frac{x}{3}\right) + C.$$

**Problem 3**. *17pts.*

(a) Determine whether $\int_0^1 \frac{dx}{2x^2+5x}$ converges and, if so, evaluate it.

(b) Determine whether $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$ converges and, if so, evaluate it.

**Solution:**

(a) We have the partial fraction decomposition:

$$\frac{1}{2x^2+5x} = \frac{\frac{1}{5}}{x} + \frac{\frac{-2}{5}}{2x+5}$$

and thus

$$\int \frac{dx}{2x^2+5x} = \frac{1}{5}\ln\left|\frac{x}{2x+5}\right| + C$$

so

$$\begin{aligned}\int_0^1 \frac{dx}{2x^2+5x} &= \lim_{R\to 0^+}\int_R^1 \frac{dx}{2x^2+5x} \\ &= \frac{1}{5}\ln\left|\frac{x}{2x+5}\right|\Bigg|_R^1 = \frac{1}{5}\ln\left|\frac{1}{7}\right| - \frac{1}{5}\ln\left|\frac{R}{2R+5}\right| \\ &= \infty.\end{aligned}$$

(b) We have

$$\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}}$$

and

$$\begin{aligned}\int_{-1}^0 \frac{dx}{\sqrt[3]{x}} &= \lim_{R\to 0^-}\int_{-1}^R \frac{dx}{\sqrt[3]{x}} = -\frac{3}{2} \\ \int_0^1 \frac{dx}{\sqrt[3]{x}} &= \lim_{R\to 0^+}\int_R^1 \frac{dx}{\sqrt[3]{x}} = \frac{3}{2}\end{aligned}$$

so

$$\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = -\frac{3}{2} + \frac{3}{2} = 0.$$

**Problem 4**. *17pts.*

Compute the surface area of revolution about the $x$-axis of $f(x) = \frac{x^2}{4} - \frac{\ln(x)}{2}$ in $[1, e]$.

**Solution:** We have:

$$f'(x) = \frac{x}{2} - \frac{1}{2x}$$

so

$$1 + (f'(x))^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$$

Hence the surface area of revolution is

$$S = 2\pi \int_1^e \left(\frac{x^2}{4} - \frac{\ln(x)}{2}\right)\left(\frac{x}{2} + \frac{1}{2x}\right) dx = \frac{\pi}{6}(e^4 - 9).$$

**Problem 5**. *17pts.*

(a) Let $T_n(x)$ be the Taylor polynomial for $f(x) = \ln(x)$ at $a = 1$. Let $c > 1$. Explain why $|\ln(c) - T_n(c)| \leq \frac{|c-1|^{n+1}}{n+1}$.

(b) Find the smallest integer value of $n$ such that $|\ln(1.5) - T_n(1.5)| \leq 10^{-2}$.

**Solution:**

(a) Computing the $k$-th derivatives of $f(x)$ we have for all integer $k$:

$$f^{(k+1)}(x) = (-1)^k \frac{k!}{x^{k+1}}$$

and thus $|f^{(k+1)}(x)| = \frac{k!}{|x|^{k+1}}$ is a decreasing function for $x > 0$. In particular, the maximum of $f^{(k+1)}(x)$ for $x$ in $[1, c]$ is $f^{(k+1)}(1) = k!$. The error bound for Taylor polynomials gives that:

$$|\ln(c) - T_n(c)| \leq K \frac{|c-1|^{n+1}}{(n+1)!}.$$

Since $K$ is a real number such that $|f^{(n+1)}(x)| \leq K$ for all $x$ in $[1, c]$, we just saw that $K = n!$, then

$$|\ln(c) - T_n(c)| \leq K \frac{|c-1|^{n+1}}{(n+1)!} \leq n! \frac{|c-1|^{n+1}}{(n+1)!} = \frac{|c-1|^{n+1}}{n+1}.$$

(b) Setting $c = 1.5$ we have

$$|\ln(1.5) - T_n(1.5)| \leq \frac{|1.5-1|^{n+1}}{n+1} = \frac{1}{2^{n+1}(n+1)}$$

and trying integer values of $n$ we find that $n = 4$ is the smallest integer such that $|\ln(1.5) - T_n(1.5)| \leq 10^{-2}$.

**Problem 6**. *17pts.*

(a) Compute the limit of the sequence with general term $a_n = \sqrt{n+3} - \sqrt{n}$.

(b) Write $\sum_{n=1}^{\infty} (\sqrt{n+3} - \sqrt{n})$ as a telescopic series and find its sum.

**Solution:**

(a) We have

$$\lim n \to \infty(\sqrt{n+3} - \sqrt{n}) = 0.$$

(b) We can write:

$$\sum_{n=1}^{\infty} (\sqrt{n+3} - \sqrt{n}) = (\sqrt{4} - \sqrt{1}) + (\sqrt{5} - \sqrt{2}) + (\sqrt{6} - \sqrt{3}) + (\sqrt{7} - \sqrt{4}) + \cdots$$

and thus from $n = 4$ onward we have that the positive terms $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, $\ldots$, will cancel the negative terms $-\sqrt{4}$, $-\sqrt{5}$, $-\sqrt{6}$, $\ldots$. The partial sums for $N > 4$ are then

$$S_N = \sum_{n=1}^{N} (\sqrt{n+3} - \sqrt{n}) = -1 - \sqrt{2} - \sqrt{3} + \sqrt{N+3}$$

whence

$$\begin{aligned}\sum_{n=1}^{\infty} (\sqrt{n+3} - \sqrt{n}) &= \lim_{N\to\infty} S_N \\ &= \lim_{N\to\infty} (-1 - \sqrt{2} - \sqrt{3} + \sqrt{N+3}) \\ &= \infty\end{aligned}$$

and the sum does not converge.