Section 7.4.: Exponential growth and decay.

Appear in Nature when something increases or decreases at a rate that is proportional to its quantity.

growth constant

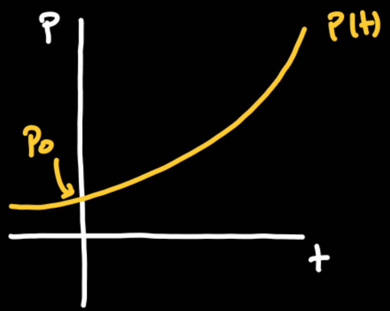
$$\boxed{P(t) = \underbrace{P_0}_{\text{initial quantity}} \cdot e^{k \cdot t}} \qquad k > 0$$

decay constant

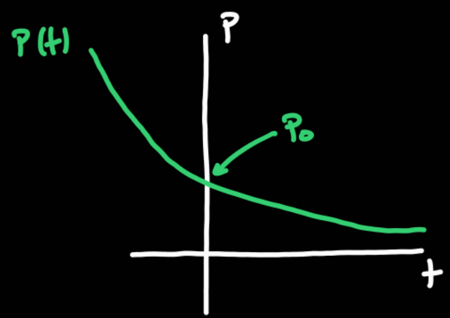
$$\boxed{P(t) = P_0 \cdot e^{-kt}}$$

$$P(0) = P_0 \cdot e^{k \cdot 0} = P_0 \cdot \underbrace{e^0}_{1} = P_0$$

Exponential growth:



Exponential decay:



To fully determine $P(t) = P_0 \cdot e^{kt}$ we need P_0 and k. Usually we are given k directly and either an initial quantity or two quantities or two different times.

k is given $\qquad\qquad\qquad\qquad$ $P(t_1) = P_1$

$P(0) = P_0 \qquad\qquad P(t_1) = P_1 \qquad\qquad P(t_2) = P_2$

Example: Population of bacteria: growth. $\quad k = 0.41 \text{ hours}^{-1} = 0.41 \dfrac{1}{\text{hours}}$.

needs to be units-less.

$P(t) = P_0 \cdot e^{\boxed{kt}}$ 1000 bacteria at $t=0$

k is given, P_0 is given.

$P(t) = 1000 \cdot e^{0.41 \cdot t}$, t in hours.

When will the population of bacteria reach 10000?

$$10000 = P(t) = 1000 \cdot e^{0.41 \cdot t} \rightsquigarrow 10 = e^{0.41 \cdot t} \rightsquigarrow \ln(10) = 0.41 \cdot t$$

$$\rightsquigarrow t = \frac{\ln(10)}{0.41}.$$

Intuitively, the derivative of a function is measuring its instantaneous rate of growth. Given a quantity $P(t)$, if $P(t)$ grows at a rate proportional to itself then:

$$\frac{d\,P(t)}{dt} = k \cdot P(t) \qquad \text{(here } k \text{ can be positive or negative)}$$

Exponential functions are <u>the only functions</u> satisfying this equation.

<u>Example:</u> We know that the population of rabbits grows at a rate proportional to the amount present.

We immediately know that $R(t) = R_0 \cdot e^{kt}$.

Find the growth constant if we begin with 1000 rabbits and after two years we have 2000 rabbits.

$$\left.\begin{array}{l} 1000 = R(0) = \underline{R_0} \cdot e^{k\cdot 0} \\ \\ 2000 = R(2) = \underline{R_0} \cdot e^{k \cdot 2} \end{array}\right\} \quad \frac{2000}{1000} = \frac{R_0 \cdot e^{k\cdot 2}}{R_0 \cdot e^{k \cdot 0}} \rightsquigarrow 2 = e^{k\cdot 2}$$

<u>Doubling time</u>: time T necessary for a population $P(t)$ to double in size.

$$P(t) = P_0 \cdot e^{k\cdot t}, \quad \underbrace{P(t+T) = 2 \cdot P(t)}_{\text{solve for } T} \qquad \boxed{T = \frac{\ln(2)}{k}}$$

<u>Half-life</u>: time T necessary for a population $P(t)$ to halve in size.

$$P(t) = P_0 \cdot e^{-kt}, \quad \underbrace{P(t+T) = \tfrac{1}{2} \cdot P(t)}_{\text{solve for } T} \qquad \boxed{T = \frac{\ln(2)}{k}}$$

<u>Section 7.1.</u>: Derivative of $f(x) = b^x$ and the number e.

<u>Exponential function</u>: $f(x) = b^x \qquad b > 0$ <u>base</u> $\qquad b \neq 1$

<u>Question</u>: What happens if $b < 0$? Try $b = -2$.

1. They are always strictly positive.

2. Their range is all positive real numbers.

3. Increasing for $b > 1$.

   Decreasing for $0 < b < 1$.

<u>Laws of exponents</u>:

Exponent zero: $b = 1.$

Products: $b^x b^y = b^{x+y}$

Quotients: $\dfrac{b^x}{b^y} = b^{x-y}$

Negative exponents: $b^{-x} = \dfrac{1}{b^x} = \left(\dfrac{1}{b}\right)^x$

Powers: $(b^x)^y = b^{xy}$

Roots: $b^{\frac{1}{n}} = \sqrt[n]{b}$ $\quad$ n natural number.

<u>Examples:</u>

1. $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$

2. $\dfrac{9^3}{3^7} = \dfrac{(3^2)^3}{3^7} = \dfrac{3^6}{3^7} = 3^{6-7} = 3^{-1} = \dfrac{1}{3}.$

<u>Derivative of the exponential function:</u> $\quad \dfrac{d\,(b^x)}{dx} = \underbrace{m}_{\text{this is } \ln(b)} \cdot b^x = \ln(b) \cdot b^x$

There is exactly one real number b such that $\ln(b) = 1.$

This number is called e.

$$\frac{d\,(e^x)}{dx} = e^x, \qquad \ln(e) = 1.$$

<u>Example:</u> Find the equation of the tangent line to $3e^x - 5x^2$ at $x = 2$.

$f(x) = 3e^x - 5x^2$

$f'(x) = 3 \cdot \dfrac{d\,(e^x)}{dx} - 5 \cdot \dfrac{d\,(x^2)}{dx} = 3e^x - 5 \cdot 2 \cdot x = 3e^x - 10x$

$f(2) = 3e^2 - 20$

$f'(2) = 3e^2 - 20$

$y = f(2) + f'(2) \cdot (x-2) = (3e^2 - 20) \cdot (1 + x - 2) = (3e^2 - 20) \cdot (x-1)$

Chain rule:

$$\boxed{\frac{d\left(e^{g(x)}\right)}{dx} = \frac{d(g(x))}{dx} \cdot e^{g(x)}}$$

Example:

$$\frac{d}{dx}\left(e^{\cos(x)}\right) = \frac{d}{dx}(\cos(x)) \cdot e^{\cos(x)} = -\sin(x) \cdot e^{\cos(x)}.$$