Recall: $f(x) = b^x \qquad b > 0 \qquad \underbrace{\frac{d}{dx}}_{\text{rate}}(f(x)) = \underbrace{\ln(b)}_{\text{constant}} \cdot \underbrace{b^x}_{\text{how much stuff we have}}$

We have: $\frac{d}{dx}(e^x) = e^x.$

Integral of exponential functions: $\qquad \frac{d}{dx}\left(e^{\overbrace{kx+b}^{\text{linear}}}\right) = k \cdot e^{kx+b}.$

$$\boxed{\int e^x \cdot dx = e^x + c_1 \quad \text{and} \quad \int e^{kx+b}\, dx = \frac{1}{k} \cdot e^{kx+b} + c_1.}$$

Derivation and integration are "inverse $\underbrace{\text{processes}}_{\text{operations}}$".

Example:

1. $$\int x \cdot e^{2x^2} \cdot dx = \underbrace{\int (e^{2x^2}) \cdot (x\,dx)}_{\text{substitution}} \overset{\downarrow}{=} \int e^u \cdot \frac{du}{4} = \frac{1}{4}\int e^u\, du =$$

$du = 4x\,dx$

if we are able to write $u = 2x^2$ and get $\int e^u\, du$, then

we can apply the integration formulas.

$$\frac{du}{dx} = 4x \rightsquigarrow du = 4x\,dx$$

$$= \frac{1}{4} e^u + c_1 \overset{\uparrow}{=} \frac{1}{4} e^{2x^2} + c_1.$$

undo the substitution $u = 2x^2$

$\int \frac{1}{y^2} = \frac{-1}{y} + c_1$

2. $$\int \frac{e^t}{1 + 2e^t + e^{2t}}\, dt = \int \frac{e^t}{(1+e^t)^2}\, dt \underset{u = e^t}{\overset{du = e^t dt}{=}} \int \frac{du}{(1+u)^2} = \frac{-1}{1+u} + c_1 =$$

$$1 + 2e^t + e^{2t} = 1 + 2y + y^2 = (1+y)^2 = (1+e^t)^2$$

$y = e^t$

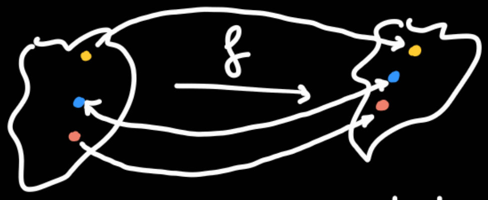
$u = e^t$

$$= \frac{-1}{1+e^t} + c_1.$$

Section 7.2: Inverse functions

Section 1.2: Inverse functions.

$f(x)$ function, the inverse of $f(x)$ undoes $f(x)$.



f has an inverse if for every input we have exactly one output and for every output there is exactly one input that is sent to this output by f.

> A function $f(x)$ with domain D and range R is invertible if and only if there is another function $g(x)$ with domain R and range D such that:
>
> $g(f(x)) = x$ for all x in D and
>
> $f(g(x)) = x$ for all x in R.

When this happens $g(x)$ is denoted $f^{-1}(x)$ and called the <u>inverse</u> of $f(x)$.

<u>How to find inverses:</u> Given a function $f(x)$. ($f(x)$ polynomial or fraction of poly.)

1. Write $y = f(x)$ and solve for x in terms of y. ($x = g(y)$)

2. Rewrite $g(x)$. (write x instead of y).

3. Check $g(f(x)) = x$ and $f(g(x)) = x$.

Example: Find the inverse of $f(x) = 2x - 18$.

1. $y = 2x - 18 \rightsquigarrow y + 18 = 2x \rightsquigarrow \frac{y+18}{2} = x \rightsquigarrow \underbrace{\frac{y}{2} + 9}_{g(y)} = x$

2. $g(y) = \frac{y}{2} + 9 \rightsquigarrow g(x) = \frac{x}{2} + 9.$

3. Check:

$$g(f(x)) = g(2x - 18) = \frac{2x - 18}{2} + 9 = \frac{2x}{2} - \frac{18}{2} + 9 = x.$$

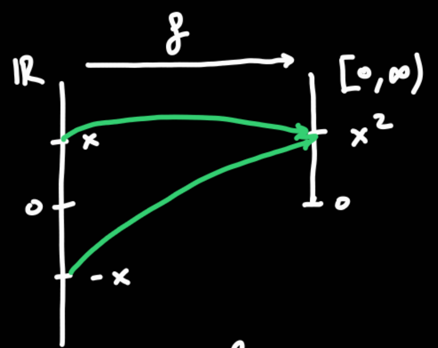
$$f(g(x)) = f\left(\frac{x}{2} + 9\right) = 2\cdot\left(\frac{x}{2} + 9\right) - 18 = \frac{2x}{2} + 2\cdot 9 - 18 = x.$$

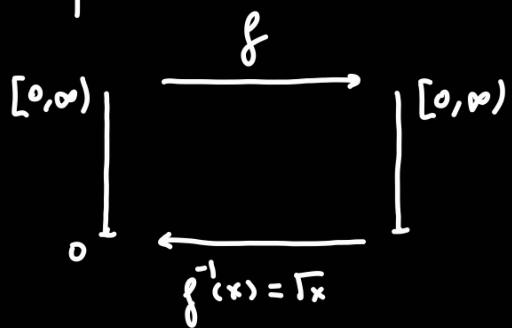
So $f^{-1}(x) = \frac{x}{2} + 9.$

Example: The function $f(x) = x^2$ does not have an inverse.

Claim: this should have inverse $g(x) = \sqrt{x}$.

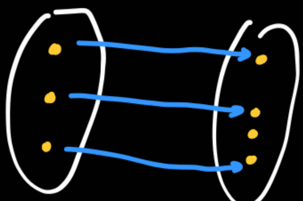
only outputs positive values



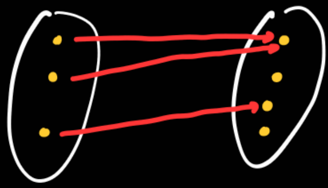


One-to-one: A function $f(x)$ is one-to-one on its domain D if for every distinct a, b in D then $f(a) \neq f(b)$.

This is equivalent to the equation $f(x) = c$ having at most one solution x in D for every c in R.

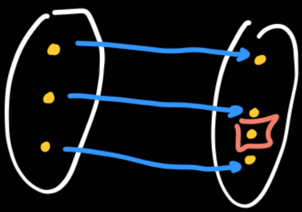


is one-to-one

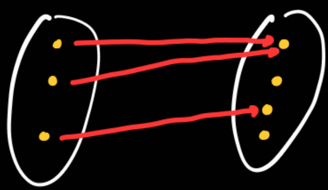


is <u>not</u> one-to-one.

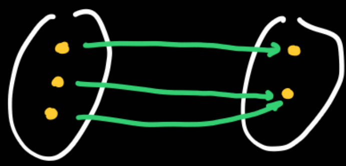
<u>Surjective:</u> for every c in R there is at least one element a in D such that $f(a) = c$. (at least one solution to $f(x) = c$)



is one-to-one
<u>not</u> surjective

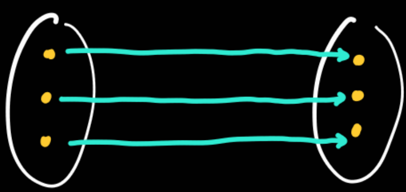


is <u>not</u> one-to-one.
<u>not</u> surjective



is surjective.
is <u>not</u> one-to-one

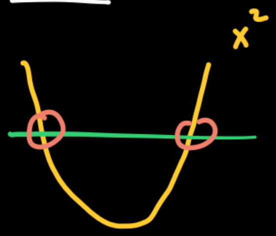
A function $f(x)$ is invertible if and only if $f(x)$ is one-to-one (injective) and surjective.



is one-to-one.
is surjective.
is invertible.

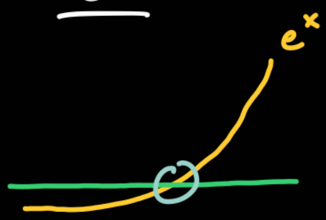
<u>Horizontal line test:</u> A function is one-to-one if and only if every horizontal line intersects the graph of the function in at most one point.

<u>x^2</u>



<u>not</u> one-to-one

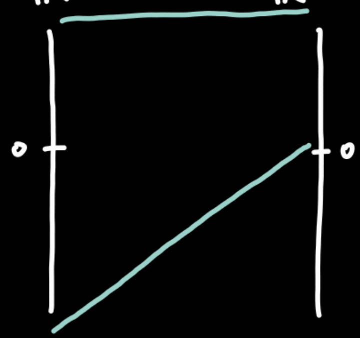
<u>e^x</u>



is one-to-one.

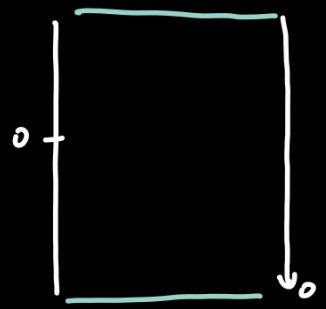
Given $f(x)$ that is one-to-one, we can always reduce the range R and make $f(x)$ an invertible function.

$\mathbb{R} \xrightarrow{e^x} \mathbb{R}$



is one-to-one
is <u>not</u> surjective

$\mathbb{R} \xrightarrow{e^x} (0, \infty)$



is one-to-one
is surjective

<u>Example:</u> Find the inverse of $f(x) = \frac{3x+2}{5x-1}$.

$D = \{ x \text{ in } \mathbb{R} \mid x \neq \frac{1}{5} \}$.

<u>1.</u> Solve $y = f(x)$ for x in terms of y.

$$y = \frac{3x+2}{5x-1} \rightsquigarrow y(5x-1) = 3x+2 \rightsquigarrow 5xy - y = 3x+2$$

$$\rightsquigarrow 5xy - 3x = y+2 \rightsquigarrow \underbrace{x(5y-3) = y+2}_{\circledast} \rightsquigarrow x = \frac{y+2}{5y-3}.$$

We can only do this when $5y-3 \neq 0$, namely $y = \frac{3}{5}$.

There is no x in $\mathcal{D}$ such that $f(x) = \frac{3}{5}$. Otherwise said $\frac{3}{5}$ is not in the range R of $f(x)$.

$$\frac{3x+2}{5x-1} = \frac{3}{5}.$$ There is no x satisfying this.

$\circledast$ $0 = \frac{3}{5} + 2$ ⚠ So y is not in R.

$g(y) = \frac{y+2}{5y-3}$ has domain $\{y \text{ in } \mathbb{R} \mid y \neq \frac{3}{5}\}$.

2. $g(x) = \frac{x+2}{5x-3}$

3. Check $g(f(x)) = x$ and $f(g(x)) = x$.

$$\underbrace{\{x \text{ in } \mathbb{R} \mid x \neq \tfrac{1}{5}\}}_{\text{domain of } f} \underset{g}{\overset{f}{\rightleftarrows}} \underbrace{\{x \text{ in } \mathbb{R} \mid x \neq \tfrac{3}{5}\}}_{\text{domain of } g}.$$

we can check that $x = \frac{1}{5}$ is not in the range of $g(x)$

we saw $x = \frac{3}{5}$ is not in the range of f

Derivative of the inverse:

derivative

$$\boxed{(f^{-1}(b))' = \frac{1}{f'(f^{-1}(b))}}$$

derivative

we evaluate at b in domain

of $f^{-1}(x)$.