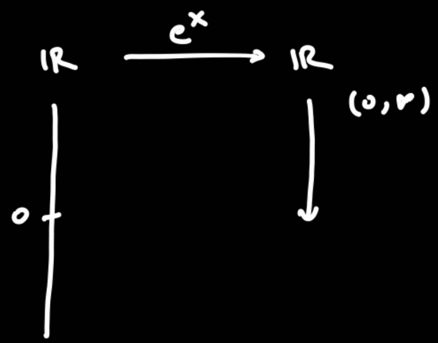
Section 7.3.: Logarithms.



$$\begin{tikzcd} \mathbb{R} \arrow[r, "e^x"] & (0,\infty) \arrow[l, dashed, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{R} & (0,\infty) \arrow[l, "\log_e(x)"'] \end{tikzcd}$$

e^x is one-to-one and every positive real number can be found as e^a for some real number a.

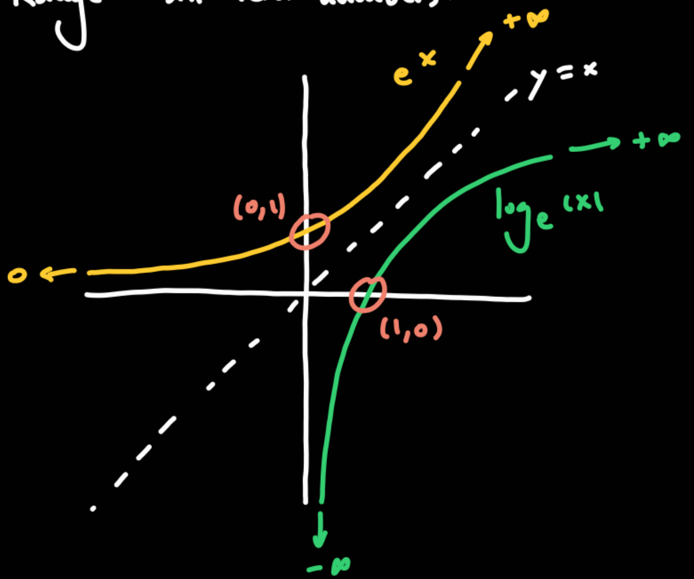
Logarithms are inverses of exponentials.

b^x will have inverse $\log_b(x)$

$b^{\log_b(x)} = x$ and $\log_b(b^x) = x$.

Domain: all positive real numbers.

Range: all real numbers.



$$\lim_{x\to+\infty} \log_b(x) = +\infty.$$

$$\lim_{x\to 0^+} \log_b(x) = -\infty.$$

Laws of logarithms:

1. Log of 1: $\log_b(1) = 0$.

2. Products: $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$

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3. Quotients: $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

4. Reciprocals: $\log_b\left(\frac{1}{y}\right) = -\log_b(y)$.

5. Powers: $\log_b(x^n) = n\cdot\log_b(x)$

Also: $\log_b(b) = 1 \iff b^1 = b$.

<u>Change of base:</u> given a base b, if we like base a, we can compute things using base a: $\quad (a = e,\ \log_e(x) = \ln(x))$

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.} \iff \boxed{\log_a(b)\cdot\log_b(x) = \log_a(x)}$$

$$\mathbb{R} \underset{\log_b(x)}{\overset{b^x}{\rightleftarrows}} (0,\infty) \qquad\qquad \mathbb{R} \underset{\frac{\log_a(x)}{\log_a(b)}}{\overset{b^x}{\rightleftarrows}} (0,\infty)$$

⟵ this will be the inverse of b^x.

To prove this equality, we just have to check:

$$b^{\frac{\log_a(x)}{\log_a(b)}} = x \quad \text{and} \quad \frac{\log_a(b^x)}{\log_a(b)} = x.$$

Now it suffices to see:

$$a^{\log_a(b)\cdot\log_b(x)} = x \quad \text{and} \quad \log_a(b)\cdot\log_b(a^x) = x.$$

$$a^{\log_a(b)\cdot\log_b(x)} = \Big(\underbrace{a^{\log_a(b)}}_{b}\Big)^{\log_b(x)} = b^{\log_b(x)} = x.$$

<u>Example:</u> $\log_6(9) + \log_6(4) = \log_6(9\cdot 4) = \log_6(36) = \log_6(6^2) = 2.$

Recall: $\underline{\frac{d}{dx}(b^x) = \ln(b)\cdot b^x}$ for all basis b. $(b>0,\ b\neq 1)$.
for all x in $\mathbb{R}$

So: $\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}}$ for base e.
for $x>0$.

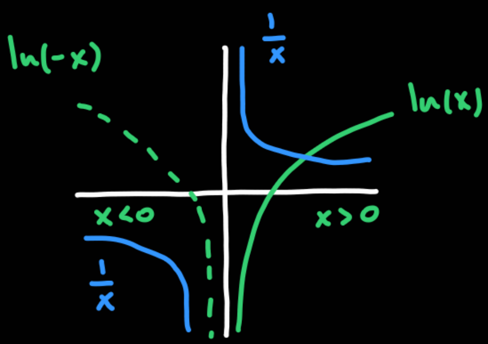
Question: What is $\frac{d}{dx}(\log_b(x))$?

Hint: Use $\log_b(x) = \frac{\ln(x)}{\ln(b)}$, so:

$$\frac{d}{dx}(\log_b(x)) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(b)}\right) = \frac{1}{\ln(b)}\cdot\frac{d}{dx}(\ln(x)) = \cdots$$

$$\underbrace{\ln(x)}_{\substack{\text{good} \\ \text{for } x>0}} \quad \overset{\frac{d}{dx}}{\underset{\int dx}{\rightleftarrows}} \quad \underbrace{\frac{1}{x}}_{\substack{\text{good} \\ \text{for } x\neq 0}} \qquad \text{for } x>0$$

$$\ln|x| \quad \overset{\frac{d}{dx}}{\underset{\int dx}{\rightleftarrows}} \quad \frac{1}{x} \qquad \text{for } x\neq 0$$



$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Example: $\frac{d}{dx}(x\cdot\ln(x)) = x\cdot\frac{d}{dx}(\ln(x)) + \frac{d(x)}{dx}\cdot\ln(x) = x\cdot\frac{1}{x} + \ln(x) = 1+\ln(x)$.

Usually we have to use the chain rule: $\boxed{\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}}$.

Example:

$$\frac{d}{dx}\left(\frac{\overbrace{(x+1)^2}^{f}\cdot\overbrace{(2x^2-3)}^{g}}{\underbrace{\sqrt{x^2+1}}_{h}}\right) = \frac{\frac{d}{dx}(f(x)\cdot g(x))\cdot h(x) - f(x)\cdot g(x)\cdot\frac{d}{dx}(h(x))}{h^2(x)} =$$

$p(x)$ $\qquad \dfrac{d\left(\frac{a}{b}\right)}{dx} = \dfrac{\frac{da}{dx}\cdot b - a\cdot\frac{db}{dx}}{b^2}$ $\qquad$ a, b real functions.

$$p(x) = \frac{(x+1)^2\cdot(2x^2-3)}{\sqrt{x^2+1}} \qquad f(x) = (x+1)^2 \qquad f'(x) = 2(x+1)$$

$$g(x) = 2x^2-3 \qquad g'(x) = 4x$$

$$h(x) = \sqrt{x^2+1} \qquad h'(x) = \frac{x}{\sqrt{x^2+1}}$$

$$= \frac{f'(x)\cdot g(x)\cdot h(x) + f(x)\cdot g'(x)\cdot h(x) - f(x)\cdot g(x)\cdot h'(x)}{h^2(x)} =$$

$$\circledast \left[\; = \frac{2(x+1)\cdot(2x^3-3)\cdot\sqrt{x^2+1} + (x+1)^2\cdot 4x\cdot\sqrt{x^2+1} - (x+1)^2\cdot(2x^3-3)\cdot\frac{x}{\sqrt{x^2+1}}}{\frac{x^2}{x^2+1}} = \right.$$

$$= (x^2+1)^{1}\cdot\underbrace{\sqrt{x^2+1}}_{(x^2+1)^{\frac{1}{2}}}\cdot\left(2(x+1)(2x^3-3) + 4x(x+1)^2 - (x+1)^2\cdot(2x^3-3)\cdot\frac{x}{x^2+1}\right)\cdot\frac{1}{x^2}$$

$$= \ldots$$

$$p(x) = \frac{(x+1)^2\cdot(2x^2-3)}{\sqrt{x^2+1}}$$, we want $p'(x)$.

<u>Logarithmic differentiation:</u> $\quad \dfrac{d}{dx}\left(\ln(p(x))\right) = \dfrac{p'(x)}{p(x)}$

So: $\quad p'(x) = p(x)\cdot\dfrac{d}{dx}\left(\ln(p(x))\right)$

$$\ln(p(x)) = \ln\left(\frac{(x+1)^2\cdot(2x^2-3)}{\sqrt{x^2+1}}\right) = \ln\left((x+1)^2\cdot(2x^2-3)\right) - \ln\left(\sqrt{x^2+1}\right) =$$

$$= \ln\left((x+1)^2\right) + \ln(2x^2-3) - \ln\left(\sqrt{x^2+1}\right) =$$

$$= 2\cdot\ln(x+1) + \ln(2x^2-3) - \frac{1}{2}\ln(x^2+1)$$

$$\frac{d}{dx}\left(\ln(p(x))\right) = 2\cdot\frac{d}{dx}\left(\ln(x+1)\right) + \frac{d}{dx}\left(\ln(2x^2-3)\right) - \frac{1}{2}\frac{d}{dx}\left(\ln(x^2+1)\right) =$$

$$= 2\cdot\frac{1}{\ldots} + \frac{4x}{\ldots} - \frac{1}{2}\cdot\frac{2x}{\ldots}$$

$$p'(x) = p(x)\cdot \frac{d}{dx}\left(\ln(p(x))\right) = \frac{(x+1)^2\cdot(2x^2-3)}{\sqrt{x^2+1}}\left(\frac{2}{x+1}+\frac{4x}{2x^2-3}-\frac{x}{x^2+1}\right) =$$

$$= \frac{1}{\sqrt{x^2+1}}\cdot\left((x+1)\cdot 2\cdot(2x^2-3) + 4x\cdot(x+1)^2 - \frac{(x+1)^2\cdot(2x^2-3)\cdot x}{x^2+1}\right) =$$

$\circledast$
$$= \underbrace{\frac{1}{(x^2+1)\sqrt{x^2+1}}}_{\frac{1}{(x^2+1)^{3/2}}}\underbrace{\left((x^2+1)\cdot(x+1)\cdot 2\cdot(2x^2-3) + (x^2+1)4x(x+1)^2 - (x+1)^2(2x^2-3)x\right)}_{\text{a polynomial}}.$$

$\circledast$ Check that they coincide.