Recall: $\frac{d}{dx}(b^x) = \ln(b)\cdot b^x$ $\qquad \mathbb{R} \underset{\ln(x)}{\overset{e^x}{\rightleftarrows}} (0,\infty)$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ $\qquad \frac{d}{dx}(\log_b(x))$ unnecessary.

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

Logarithmic differentiation:

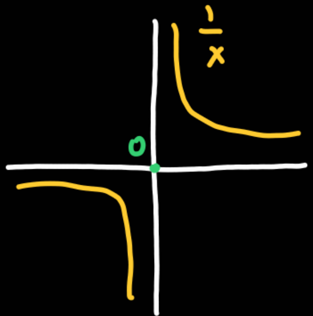
$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$ if $f(x)$ has many multiplications in both the numerator and denominator, $\ln(f(x))$ will be easy.

$$f'(x) = \frac{d}{dx}(\ln(f(x)))\cdot f(x)$$

Section 7.7.: L'Hôpital's rule.

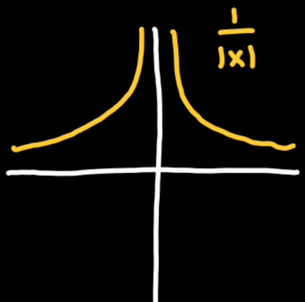
$\frac{1}{x}$ what is the value at $x=0$?

what is the function approaching when x goes to 0?



if we approach from the right we go to $+\infty$.

if we approach from the left we go to $-\infty$.



if we approach $x=0$ we go to $+\infty$.

If limits from left and right coincide, then we say that the limit exists.

<u>L'Hôpital's rule:</u> If the limit exists, then: $\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.}$

$f(x)$, $g(x)$ have to be differentiable.

$f(a) = 0 = g(a)$ and $g'(x) \neq 0$ except maybe at $x = a$.

This is <u>not</u> true without some conditions! There are functions $f(x)$ and $g(x)$

that do <u>not</u> satisfy this. $g(x) = x$ Find some!

$\frac{1}{x} \xrightarrow[x\to 0]{} +\infty$ $\frac{x}{1} \xrightarrow[x\to 0]{} 0$

$\frac{f(x)}{g(x)} \xrightarrow[x\to 0]{} \frac{0}{0}$ $\frac{\frac{1}{f(x)}}{\frac{1}{g(x)}} \xrightarrow[x\to 0]{} \frac{+\infty}{+\infty}$

<u>Example:</u> Evaluate:

$$\lim_{x\to 2} \frac{x^3-8}{x^4+2x-20} = \lim_{x\to 2} \frac{f'(x)}{g'(x)} = \lim_{x\to 2} \frac{3x^2}{4x^3+2} = \frac{3\cdot 4}{4\cdot 8+2} = \frac{12}{34} = \frac{6}{17}.$$

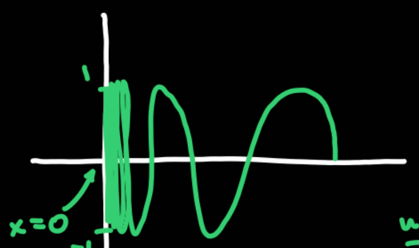
LHR

How?

$\frac{f(x)}{g(x)}$ $f(x) = x^3-8$ $f(2) = 0 = g(2)$

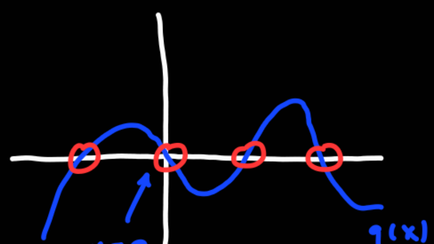
$g(x) = x^4+2x-20$

$g'(x) = 4x^3+2$ is <u>not</u> zero near $x = 2$.



$g(x) = \sin\left(\frac{1}{x}\right)$

<u>not</u> acceptable



Example: Evaluate:

$$\lim_{x\to\frac{\pi}{2}} \frac{\cos^2(x)}{1-\sin(x)} \underset{\uparrow \text{LHR}}{=} \frac{0}{0} = \lim_{x\to\frac{\pi}{2}} \frac{2\cdot(\cancel{-}\sin(x))\cdot\cancel{\cos(x)}}{\cancel{-}\cancel{\cos(x)}} \underset{\uparrow \text{simplify}}{=} \lim_{x\to\frac{\pi}{2}} 2\cdot\sin(x) = 2.$$

$f(x) = \cos^2(x)$ $\quad f\left(\frac{\pi}{2}\right) = 0$

$g(x) = 1-\sin(x)$ $\quad g\left(\frac{\pi}{2}\right) = 1-1 = 0$ $\quad g'(x) = -\cos(x)$

Example: Evaluate:

$$\lim_{x\to 0^+} x\cdot\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \underset{\uparrow \text{LHR}}{=} \frac{-\infty}{+\infty} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

$x = \dfrac{1}{\frac{1}{x}}$

$f(x) = \ln(x) \xrightarrow{x\to 0^+} -\infty$

$g(x) = \frac{1}{x} \xrightarrow[x\to 0^+]{} +\infty$ $\quad g'(x) = \frac{-1}{x^2}$

functions can approach $\pm\infty$ at $x = 0^+$.

Example: Evaluate:

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos(x)-1} \underset{\uparrow \text{LHR}}{=} \frac{0}{0} = \lim_{x\to 0} \frac{e^x-1}{-\sin(x)} \underset{\uparrow \text{LHR}}{=} \frac{0}{0} = \lim_{x\to 0} \frac{e^x}{-\cos(x)} = \frac{1}{-1} = -1.$$

$f(x) = e^x - x - 1$ $\quad f(0) = 1-0-1 = 0$

$g(x) = \cos(x) - 1$ $\quad g(0) = 1-1 = 0$ $\quad g'(x) = -\sin(x)$

$f(x) = e^x - 1$ $\quad f(0) = 1-1 = 0$

$g(x) = -\sin(x)$ $\quad g(0) = 0$ $\quad g'(x) = -\cos(x)$

Example: Evaluate:

$$\lim_{x\to 0}\left(\underbrace{\frac{1}{\sin(x)}-\frac{1}{x}}_{\substack{\frac{1}{0}-\frac{1}{0} \\ \infty-\infty}}\right)=\lim_{x\to 0}\frac{x-\sin(x)}{x\cdot\sin(x)}\underset{\text{LHR}}{=}\frac{0}{0}=\lim_{x\to 0}\frac{1-\cos(x)}{x\cdot\cos(x)+\sin(x)}\underset{\text{LHR}}{=}\frac{0}{0}$$

$f(x)=x-\sin(x)$

$g(x)=x\cdot\sin(x)$

$$\frac{1}{\sin(x)}-\frac{1}{x}=\frac{x}{x}\cdot\frac{1}{\sin(x)}-\frac{\sin(x)}{\sin(x)}\cdot\frac{1}{x}=\frac{x-\sin(x)}{x\cdot\sin(x)}.$$

$$=\lim_{x\to 0}\frac{\sin(x)}{-x\cdot\sin(x)+2\cos(x)}=0.$$

$f(x)=1-\cos(x)$

$g(x)=x\cdot\cos(x)+\sin(x)$

If your function $f(g(x))$ is nice then: $\longrightarrow$ continuous

f is the nice one.

$$\lim_{x\to a} f(g(x))=f\left(\lim_{x\to a} g(x)\right).$$

$$\lim_{x\to a} e^{g(x)}=e^{\lim_{x\to a} g(x)}.$$

$f(x)=e^x$

Example: Evaluate:

$$\lim_{x\to 0^+} x^x \underset{\uparrow}{=} \lim_{x\to 0^+} e^{\ln(x^x)} \underset{\uparrow}{=} \lim_{x\to 0^+} e^{x\cdot\ln(x)}=e^{\lim_{x\to 0^+} x\cdot\ln(x)} \underset{\uparrow}{=} e^0=1.$$

$x^x=e^{\ln(x^x)}$

$\ln(x^x)=x\cdot\ln(x)$

$\lim_{x\to 0^+} x\cdot\ln(x)=0$