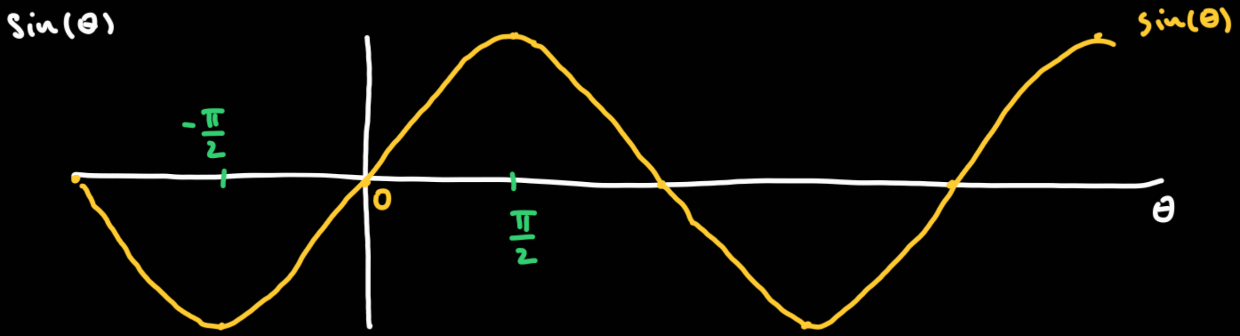
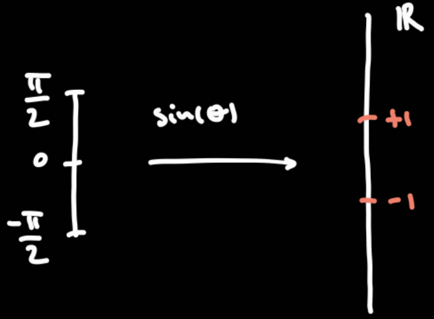
Recall: LHR.

Section 7.8.: Inverse trigonometric functions.



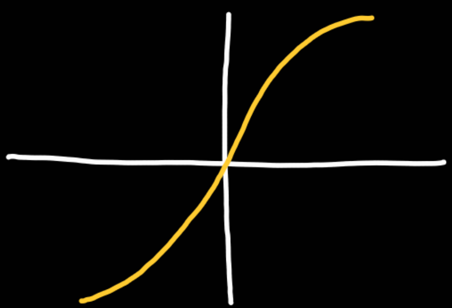
The function $\sin(\theta)$ is one-to-one on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

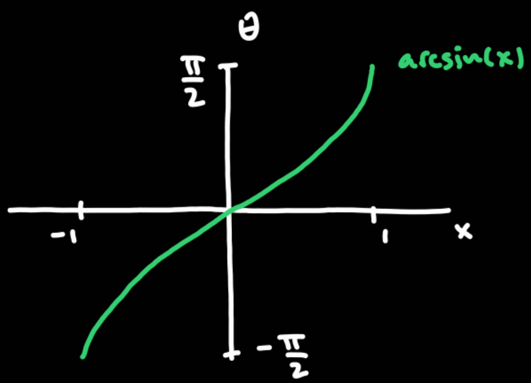


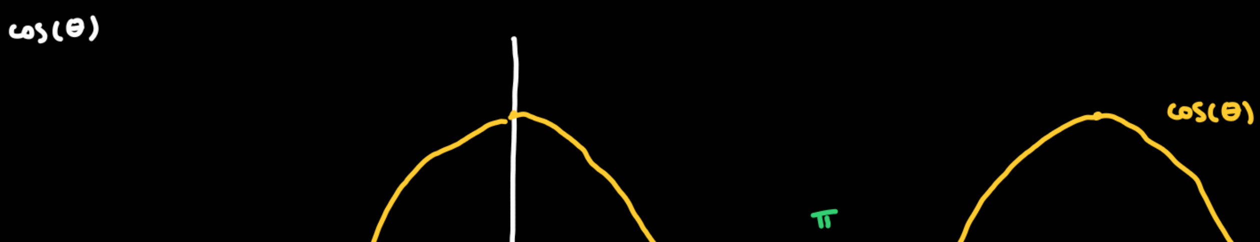
The function $\sin(\theta)$ with domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and range $[-1, 1]$ is invertible.

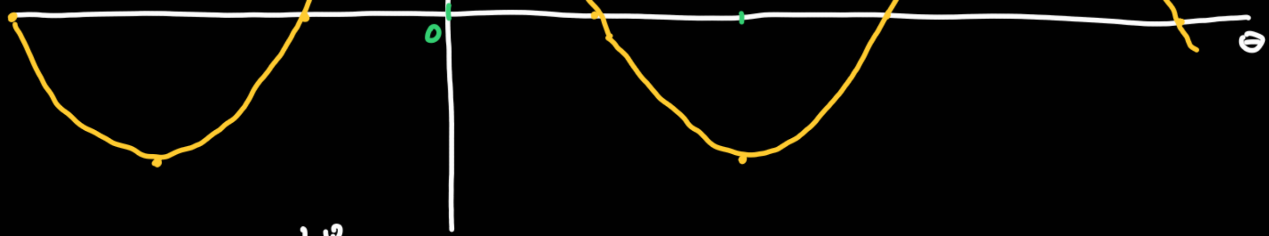
The inverse is the arcsine: $\boxed{\arcsin(x).}$

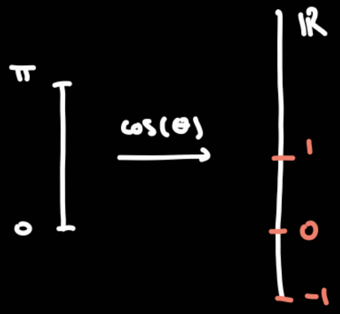
$\sin(\arcsin(x)) = x \qquad \arcsin(\sin(\theta)) = \theta.$

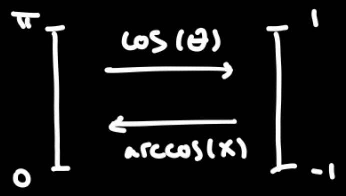


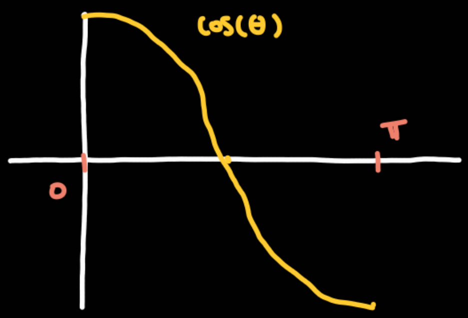


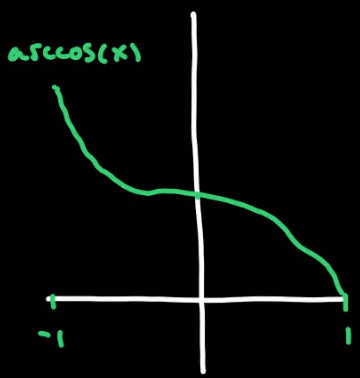












The function cosine with domain $[0, \pi]$ and range $[-1, 1]$ is invertible,

we call the inverse $\boxed{\arccos(x)}$, it has domain $[-1, 1]$ and range $[0, \pi]$.

Derivatives of inverse trig. functions:

$$\boxed{\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\arccos(x)\right) = -\frac{1}{\sqrt{1-x^2}}}$$

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

Function:	Inverse:	Derivative:
$\sin(\theta)$	$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos(\theta)$	$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}}$

$\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$	$\arctan(x)$	$\dfrac{1}{x^2+1}$
$\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$	$\text{arccot}(x)$	$\dfrac{-1}{x^2+1}$
$\sec(\theta) = \dfrac{1}{\cos(\theta)}$	$\text{arcsec}(x)$	$\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\csc(\theta) = \dfrac{1}{\sin(\theta)}$	$\text{arccsc}(x)$	$\dfrac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

Example: Evaluate:

$$\int_{-\frac{3}{4}}^{0} \frac{dx}{\underbrace{\sqrt{9-16x^2}}_{1-x^2}} = \int_{-\frac{3}{4}}^{0} \frac{1}{3}\cdot\frac{dx}{\sqrt{1-\underbrace{\left(\frac{4x}{3}\right)}_{u}^2}} = \int_{-1}^{0} \frac{1}{3}\cdot\frac{\frac{3}{4}\,du}{\sqrt{1-u^2}} =$$

This looks like arcsin or arccos.

$$u = \frac{4x}{3} \qquad u(0) = 0 \qquad u\left(\frac{-3}{4}\right) = -1$$

$$du = \frac{4}{3}\,dx$$

$$\sqrt{9-16x^2} = \sqrt{9-\frac{9\cdot 16}{9}x^2} = \sqrt{9\cdot\left(1-\frac{16}{9}x^2\right)} = 3\sqrt{1-\frac{4^2x^2}{3^2}} =$$

$$= 3\sqrt{1-\left(\frac{4x}{3}\right)^2}$$

$$= \int_{-1}^{0} \frac{1}{4}\cdot\underbrace{\frac{du}{\sqrt{1-u^2}}}_{\text{arcsin}} = \frac{1}{4}\cdot\arcsin(u)\Big|_{-1}^{0} =$$

$$= \frac{1}{4}\cdot\left(\underbrace{\arcsin(0)}\, - \underbrace{\arcsin(-1)}\right) = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}.$$

$$\sin\Big(\underbrace{\arcsin(0)}_{0}\Big) = 0 \qquad \sin\Big(\underbrace{\arcsin(-1)}_{-\frac{\pi}{2}}\Big) = -1$$

Section 7.9.: Hyperbolic functions.

Function:	Derivatives:	Hyperbolic:	Derivatives:

$e^x - e^{-x}$ $\qquad$ $e^x + e^{-x}$

$\sin(\theta)$	$\cos(\theta)$	$\sinh(x) = \frac{\cdots}{2}$	$\frac{e^x + e^{-x}}{2} = \cosh(x)$
$\cos(\theta)$	$-\sin(\theta)$	$\cosh(x) = \frac{e^x + e^{-x}}{2}$	$\sinh(x)$
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\sec^2(\theta)$	$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$\operatorname{sech}^2(x)$
$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$	$-\csc^2(\theta)$	$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$	$-\operatorname{csch}^2(x)$
$\sec(\theta) = \frac{1}{\cos(\theta)}$	$\sec(\theta)\cdot\tan(\theta)$	$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$	$-\operatorname{sech}(x)\cdot\tanh(x)$
$\csc(\theta) = \frac{1}{\sin(\theta)}$	$-\csc(\theta)\cdot\cot(\theta)$	$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$	$-\operatorname{csch}(x)\cdot\coth(x)$

Inverses of hyperbolic functions:

Function:	Inverse:	Domain:	Derivative:
$\sinh(x)$	$\operatorname{arcsinh}(x)$	$\mathbb{R}$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh(x)$	$\operatorname{arccosh}(x)$	$[1,\infty)$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh(x)$	$\operatorname{arctanh}(x)$	$(-1,1)$	$\frac{1}{1-x^2}$
$\coth(x)$	$\operatorname{arccoth}(x)$	$(-\infty,-1)\cup(1,\infty)$	$\frac{1}{1-x^2}$
$\operatorname{sech}(x)$	$\operatorname{arcsech}(x)$	$(0,1]$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{csch}(x)$	$\operatorname{arccsch}(x)$	$(-\infty,0)\cup(0,\infty)$	$\frac{-1}{\lvert x\rvert\sqrt{x^2+1}}$

Example: Compute:

$$\frac{d}{dx}\left(\operatorname{arctanh}(x)\right) = \frac{1}{\operatorname{sech}^2(\operatorname{arctanh}(x))} = \frac{1}{1-x^2}.$$

We know how to compute the differential of inverses: if $g(x)$ is the inverse of $f(x)$ then $g'(x) = \frac{1}{f'(g(x))}$

$f(x) = \tanh(x)$ $\quad f'(x) = \operatorname{sech}^2(x)$

$g(x) = \operatorname{arctanh}(x)$

Recall: $\sin^2(\theta) + \cos^2(\theta) = 1$

$1 = \cosh^2(t) - \sinh^2(t)$

$\frac{1}{\cosh^2(t)} = 1 - \frac{\sinh^2(t)}{\cosh^2(t)}$

$\operatorname{sech}^2(t) = 1 - \tanh^2(t)$

$t = \operatorname{arctanh}(x)$

$\operatorname{sech}^2(\operatorname{arctanh}(x)) = 1 - x^2$