Section 8.5.: The method of partial fractions.

$$f(x) = \frac{P(x)}{Q(x)}$$

1. $P(x)$ has higher degree than $Q(x)$. Do the division of polynomials.

2. $Q(x)$ has higher degree than $P(x)$.

Example: Integrate:

$$\int \frac{dx}{x^2-1} = \int \frac{dx}{(x-1)(x+1)} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{\ln|x-1|}{2} - \frac{\ln|x+1|}{2}.$$

$x^2-1=(x-1)(x+1)$

$$\frac{1}{(x-1)(x+1)} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

Given a polynomial with real entries, it can always be factored into terms of degree 2 and terms of degree 1.

Example:
$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \underbrace{\overset{1}{\frac{B}{x+2}} + \overset{2}{\frac{C}{(x+2)^2}}}$$

$$\frac{1}{(x-1)(x+2)^3} = \frac{A}{x-1} + \underbrace{\overset{1}{\frac{B}{x+2}} + \overset{2}{\frac{C}{(x+2)^2}} + \overset{3}{\frac{D}{(x+2)^3}}}$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \underbrace{\overset{1}{\frac{B}{x^2+1}} + \overset{2}{\frac{C}{(x^2+1)^2}}}$$

To compute $A, B, C, D, \dots,$ we expand the equality.

Example: Find A, B, C:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$\Big\downarrow$ multiply by $(x-1)(x+2)^2$

$$1 = A\cdot(x+2)^2 + B\cdot(x-1)(x+2) + C\cdot(x-1) \quad \longleftarrow$$ this is an equality for all real numbers x.

Set $x=1$ then: $1 = A\cdot 9 + 0 + 0$ so $A = \frac{1}{9}$.

Set $x=-2$ then: $1 = 0 + 0 + C\cdot(-3)$ so $C = \frac{-1}{3}$.

Set $x=0$ then: $1 = A\cdot 4 + B\cdot(-2) + C\cdot(-1) = \frac{4}{9} - 2\cdot B + \frac{1}{3} =$

$= \frac{4}{9} + \frac{3}{9} - 2\cdot B = \frac{7}{9} - 2\cdot B$ so $B = \frac{7}{2\cdot 9} - \frac{1}{2} =$

$= \frac{7}{18} - \frac{9}{18} = \frac{-2}{18} = \frac{-1}{9}$.

$$\frac{1}{(x-1)(x+2)^2} = \frac{\frac{1}{9}}{x-1} + \frac{\frac{-1}{3}}{x+2} + \frac{\frac{-1}{9}}{(x+2)^2}$$

Exercise: Decompose into partial fractions:

$$\frac{5x^2+x-28}{x^3-4x^2+x+6} = \frac{-2}{x+1} + \frac{2}{x-2} + \frac{5}{x-3}$$

Example: Decompose into partial fractions and integrate: $\frac{x^2+2}{2x^3-6x^2-12x+16}$.

$$2x^3-6x^2-12x+16 = (x-1)(x+2)(2x-8)$$

$$\frac{x^2+2}{2x^3-6x^2-12x+16} = \frac{A}{x-1} + \frac{B}{2x-8} + \frac{C}{x+2} = \frac{-1}{6(x-1)} + \frac{1}{2x-8} + \frac{1}{6(x+2)}.$$

$$x^2+2 = A\cdot(x+2)(2x-8) + B\cdot(x-1)(x+2) + C\cdot(x-1)(2x-8).$$

$$x=1 \rightsquigarrow A = \frac{-1}{6}.$$

$$x=4 \rightsquigarrow B = 1.$$

$$x=-2 \rightsquigarrow C = \frac{1}{6}.$$

So:
$$\int \frac{x^2+2}{2x^3-6x^2-12x+16}\,dx = \int\left(\frac{-1}{6(x-1)} + \frac{1}{2x-8} + \frac{1}{6(x+2)}\right)dx =$$

$$= -\int \frac{dx}{6(x-1)} + \int \frac{dx}{2x-8} + \int \frac{dx}{6(x+2)} = \frac{-\ln|x-1|}{6} + \frac{\ln|x-4|}{2} + \frac{\ln|x+2|}{6} + C.$$

Example: Find the partial fraction decomposition and integrate:

$$\frac{4-x}{x(x^2+2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

Exercise!

$A=1 \qquad B=-1 \qquad C=0 \qquad D=-2 \qquad E=-1$

We can integrate:

$$\int \frac{4x\,dx}{x(x^2+2)^2} = \underbrace{\int \frac{1}{x}\,dx}_{\ln|x|} - \underbrace{\int \frac{x}{x^2+2}\,dx}_{\frac{\ln|x^2+2|}{2}} - \int \frac{2x+1}{(x^2+2)^2}\,dx$$

$$\int \frac{2x+1}{(x^2+2)^2}\,dx = \underbrace{\int \frac{2x\,dx}{(x^2+2)^2}}_{u=x^2+2} + \underbrace{\int \frac{dx}{(x^2+2)^2}}_{\text{tricky!}}$$

$$\int \frac{1}{u^2}\,du = \frac{-1}{u}$$

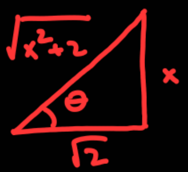
$$\frac{-1}{x^2+2}$$

$$\int \frac{dx}{(x^2+2)^2} = \int \frac{\sqrt{2}\cdot \sec^2(\theta)\, d\theta}{\left(2\cdot \sec^2(\theta)\right)^2} = \int \frac{\sqrt{2}\cdot \sec^2(\theta)\, d\theta}{4\cdot \sec^4(\theta)} = \frac{\sqrt{2}}{4}\int \cos^2(\theta)\, d\theta =$$

$$\theta = \arctan\left(\frac{x}{\sqrt{2}}\right)$$

$$\cos(\theta) = \frac{\sqrt{2}}{\sqrt{x^2+2}}$$

$$\sin(\theta) = \frac{x}{\sqrt{x^2+2}}$$



$$\tan(\theta) = \frac{x}{\sqrt{2}} \qquad \sqrt{2}\cdot \tan(\theta) = x \qquad dx = \sqrt{2}\cdot \sec^2(\theta)\, d\theta$$

$$x^2+2 = 2\cdot \tan^2(\theta) + 2 = 2\cdot \underbrace{\left(\tan^2(\theta)+1\right)}_{\sec^2(\theta)} = 2\cdot \sec^2(\theta)$$

integration by parts

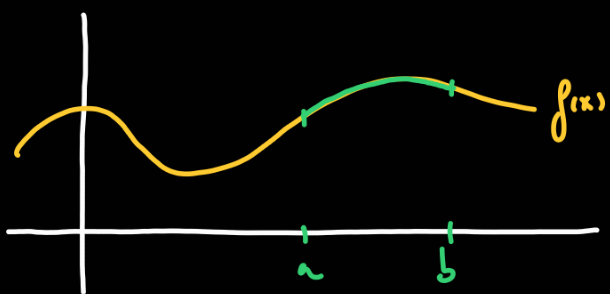
$$= \frac{\sqrt{2}}{4}\cdot\left(\frac{\theta}{2} + \frac{\cos(\theta)\cdot \sin(\theta)}{2}\right) =$$

$$= \frac{\sqrt{2}}{8}\cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8}\cdot \frac{\sqrt{2}}{\sqrt{x^2+2}}\cdot \frac{x}{\sqrt{x^2+2}} =$$

$$= \frac{\sqrt{2}}{8}\cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{4}\cdot \frac{x}{x^2+2} + C$$

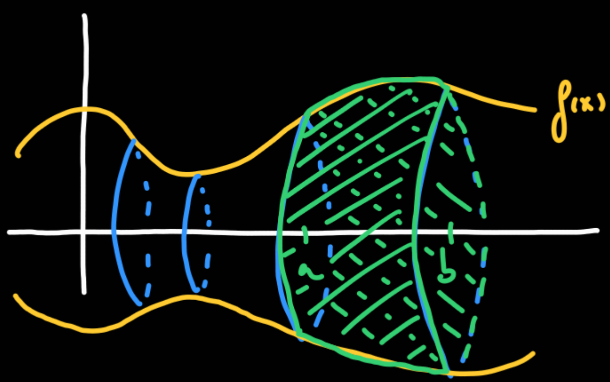
$$\int \frac{4x\, dx}{x\,(x^2+2)^2} = \ln|x| - \frac{\ln|x^2+2|}{2} + \frac{1}{x^2+2} - \frac{\sqrt{2}}{8}\cdot \arctan\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{4}\cdot \frac{x}{x^2+2} + C.$$

Section 9.1.: Arc-length and surface area.



$$\boxed{\int_a^b \sqrt{1+(f'(x))^2}\, dx}$$

Arc-length.



Surface area.

$$\boxed{\int_a^b 2\pi\, f(x)\cdot \sqrt{1+(f'(x))^2}\, dx}$$