Recall: $\dfrac{P(x)}{Q(x)}$

1. If deg. $P(x)$ is larger than deg. $Q(x)$ we do the long division of polynomials:

$$P(x) = C(x)\cdot Q(x) + \underbrace{R(x)}_{\text{deg. } R(x) \text{ less than deg. } Q(x)}$$

$$\frac{P(x)}{Q(x)} = C(x) + \frac{R(x)}{Q(x)}$$

2. If deg. $P(x)$ is less than deg. $Q(x)$.

$$x\cdot(x+1)^2 \cdot \quad \frac{1}{x\cdot(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \quad \cdot\, x\cdot(x+1)^2$$

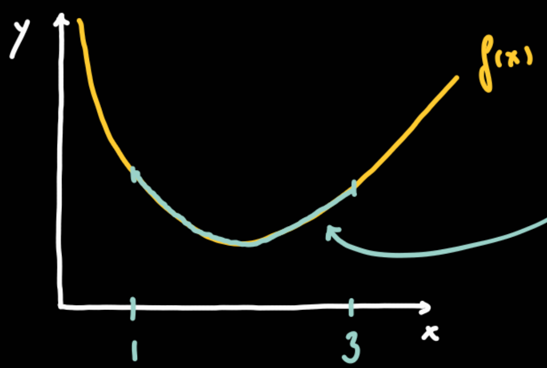
$$\frac{1}{x\cdot(x^2+1)^3} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} + \frac{Fx+G}{(x^2+1)^3}$$

Section 9.1: Arc-length and surface area.

arc-length: $\displaystyle\int_a^b \sqrt{1+(f'(x))^2}\,dx$

surface area: $\displaystyle 2\pi\int_a^b f(x)\cdot\sqrt{1+(f'(x))^2}\,dx$

Example: Find the arc-length of $f(x) = \dfrac{x^3}{12} + \dfrac{1}{x}$ in $[1,3]$.



$$\int_1^3 \sqrt{1+(f'(x))^2}\,dx$$

$\cos^2\theta + \sin^2\theta = 1$

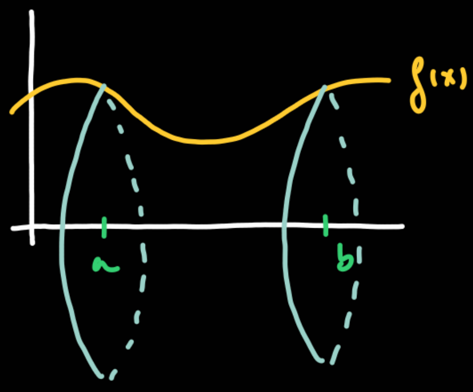
$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2} \qquad (f'(x))^2 = \frac{x^4}{16} + \frac{1}{x^4} - \frac{1}{2}$$

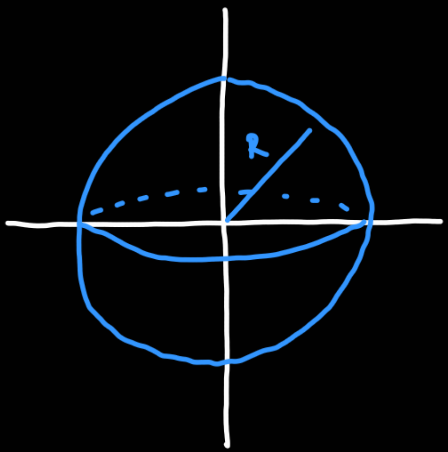
$$1+(f'(x))^2 = 1 + \frac{x^4}{16} + \frac{1}{x^4} - \frac{1}{2} = \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

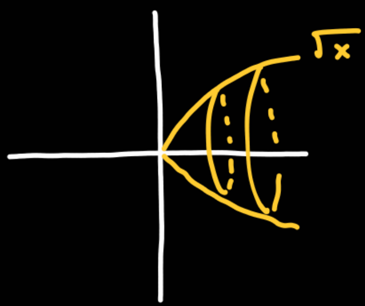
$$\int_1^3 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \left(\frac{x^3}{12} - \frac{1}{x}\right)\Big|_1^3 = \left(\frac{9}{12} - \frac{1}{3}\right) - \left(\frac{1}{12} - 1\right) = \frac{17}{6}.$$

<u>Example:</u> Calculate the surface area of a sphere of radius R.

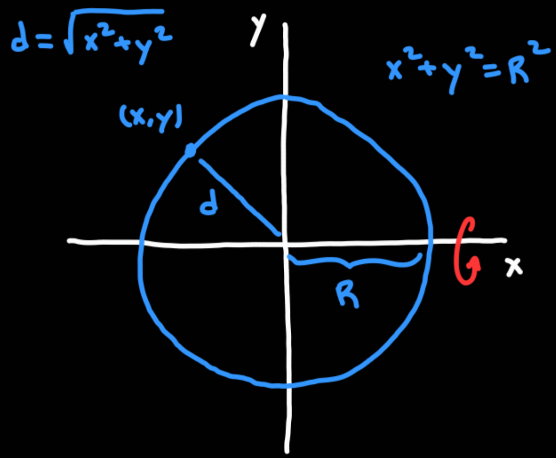
$$\int_a^b 2\pi\, f(x) \cdot \sqrt{1 + (f'(x))^2}\, dx$$

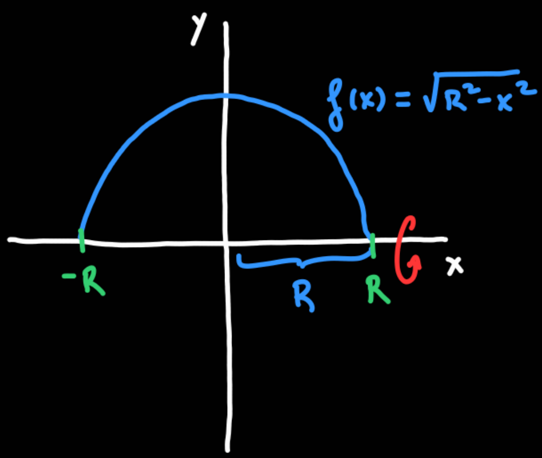






spherical hyperbolbid



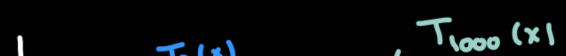


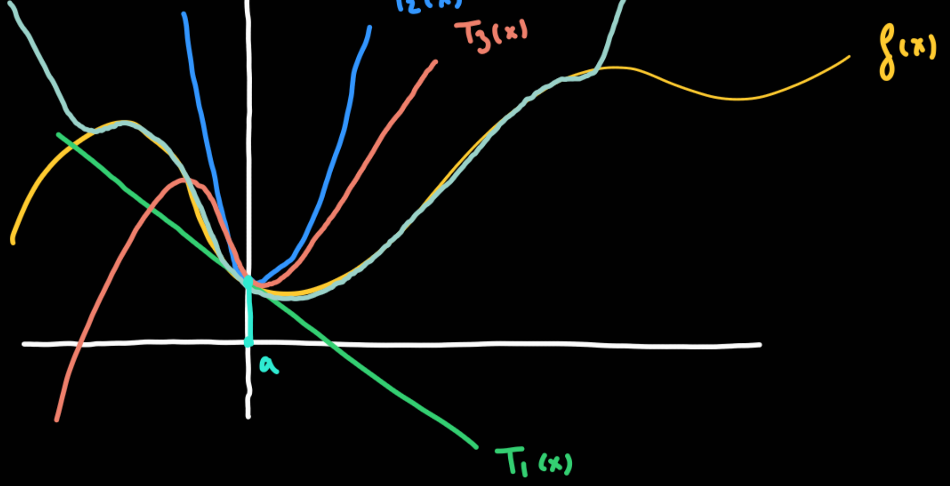
$$\int_{-R}^{R} 2\pi\, f(x) \cdot \sqrt{1 + (f'(x))^2}\, dx = 4\pi R^2$$

$$f'(x) = \frac{-x}{\sqrt{R^2 - x^2}} \qquad 1 + (f'(x))^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}.$$

$$\int_{-R}^{R} 2\pi \cdot \sqrt{R^2 - x^2} \cdot \sqrt{\frac{R^2}{R^2 - x^2}}\, dx = 2\pi R \int_{-R}^{R} dx = 2\pi R \cdot x \Big|_{-R}^{R} = 2\pi R\big(R - (-R)\big) = 4\pi R^2.$$

<u>Section 9.4.:</u> Taylor polynomials.





T_1 uses information about the first derivative. $\quad T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$

T_2 second $\quad T_2(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$

T_3 third $\quad T_3(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}\cdot(x-a)^3$

T_{1000} 1000-th

<u>Example:</u> Compute the third and fourth Maclaurin polynomials of $f(x) = e^x$.

Note that $f^{(n)}(x) = e^x$ for all natural number n.

So: $\quad f(0) = f^{(1)}(0) = f^{(2)}(0) = f^{(3)}(0) = f^{(4)}(0) = 1.$

So:

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$\boxed{T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j = \underbrace{f(a)}_{j=0} + \underbrace{\frac{f^{(1)}(a)}{1!}(x-a)}_{j=1} + \cdots + \underbrace{\frac{f^{(n)}(a)}{n!}(x-a)^n}_{j=n}}$$

$f(x)$	a	polynomial
e^x	0	$T_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$
$\sin(x)$	0	$T_{2n+1}(x) = T_{2n+2}(x) = \sum_{j=0}^{n} (-1)^j \cdot \frac{x^{2j+1}}{(2j+1)!}$
$\cos(x)$	0	$T_{2n}(x) = T_{2n+1}(x) = \sum_{j=0}^{n} (-1)^j \cdot \frac{x^{2j}}{(2j)!}$
$\ln(x)$	1	$T_n(x) = \sum_{j=1}^{n} (-1)^{j-1} \cdot \frac{(x-1)^j}{j}$
$\frac{1}{1-x}$	0	$T_n(x) = \sum_{j=0}^{n} x^j$

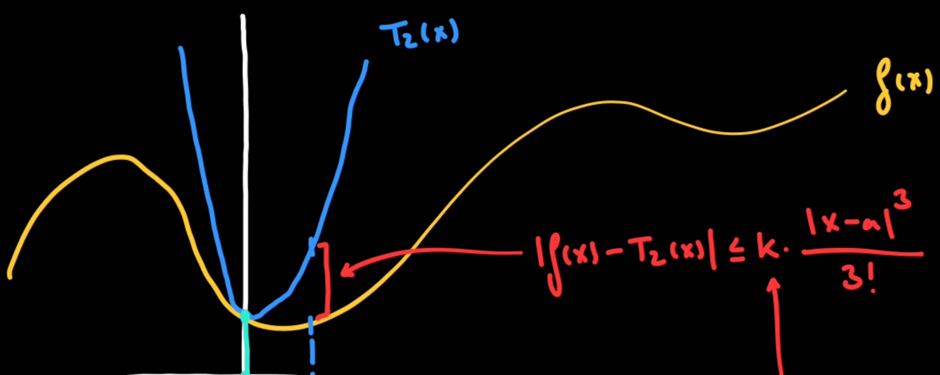
Error bound:

$$\boxed{|f(x) - T_n(x)| \leq k \cdot \frac{|x-a|^{n+1}}{(n+1)!}}$$

$\underbrace{|f^{(n+1)}(u)| \leq k}$ for all u between x and a. k fixed real number.

$f^{(n+1)}(x)$ is continuous.

The error between $f(x)$ and $\underbrace{T_n(x)}_{\text{approximation of } f(x) \text{ up to } n\text{-th derivative.}}$ is controlled by the $\underbrace{(n+1)\text{-th derivative.}}_{K}$



a $\quad x$

$|f^{(3)}(u)| \leq K$ depends on the 3rd derivative

for u between 0 and x.

Example: Let $f(x) = \cos(x)$. Find an integer n such that the n-th Maclaurin ($a=0$) polynomial $T_n(x)$ has error less than 10^{-5} at $x = 0.2$.

Step 1: Find K.

We know that $|f^{(n)}(x)| = |\cos(x)|$ or $|f^{(n)}(x)| = |\sin(x)|$. We always have $|f^{(n)}(x)| \leq 1$. So take $K=1$, now $|f^{(n)}(x)| \leq 1 = K$.

Step 2: Find n using the error bound.

$$|\cos(0.2) - T_n(0.2)| \leq K \cdot \frac{|x-a|^{n+1}}{(n+1)!} = 1 \cdot \frac{|0.2-0|^{n+1}}{(n+1)!} = \frac{0.2^{n+1}}{(n+1)!} < 10^{-5}$$

we want this, we find n using this.

$n=1$

$0.02 \overset{?}{<} 10^{-5}$

No.

$n=2$

$\frac{1}{750} \overset{?}{<} 10^{-5}$

No.

$n=3$

$\frac{1}{15000} \overset{?}{<} 10^{-5}$

No.

$n=4$

$\frac{1}{375000} \overset{?}{<} 10^{-5}$

YES!

We have that $|f(0.2) - T_4(0.2)| < 10^{-5}$, so we were looking for $n=4$.