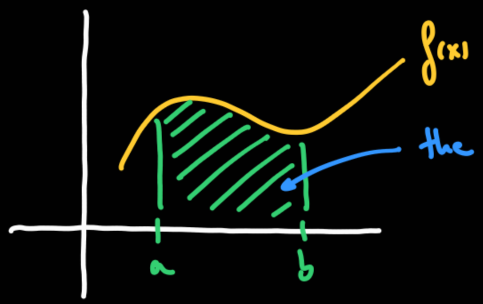
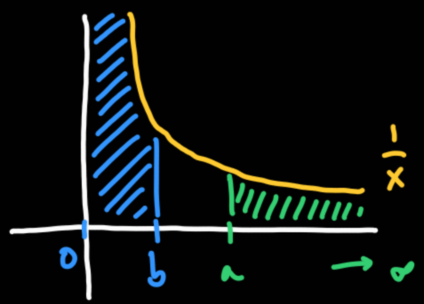
Section 8.6.: Improper integrals.

These are integrals where infinity appears.

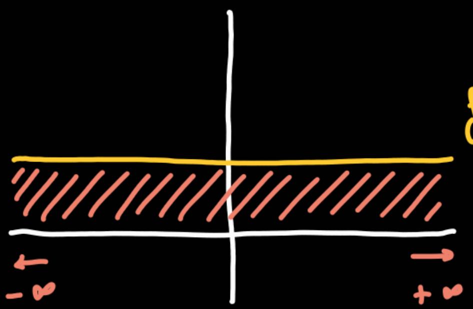


the area is $\int_a^b f(x)\,dx$

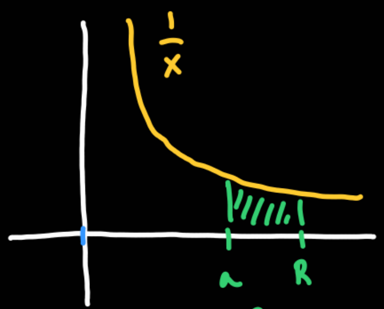


$\int_0^b \frac{1}{x}\,dx$ is it finite?

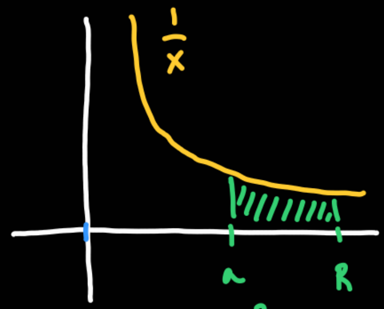
$\int_a^\infty \frac{1}{x}\,dx$ is it finite?



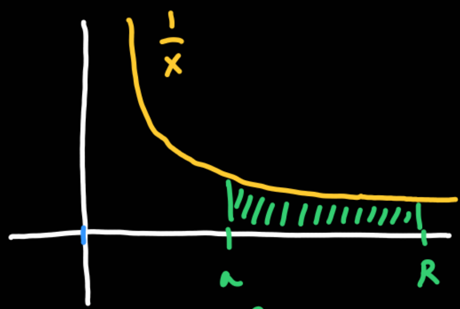
$\int_{-\infty}^{\infty} f(x)\,dx$ is infinite (i.e. diverges).



$\int_a^R f(x)\,dx$



$\int_a^R f(x)\,dx$



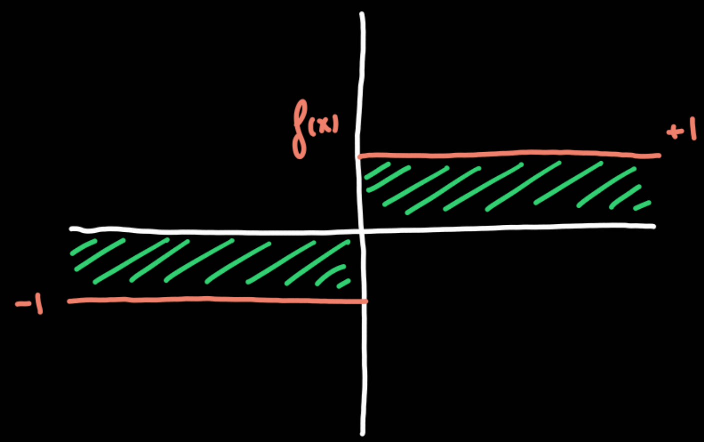
$\int_a^R f(x)\,dx$

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx}$$

$$\boxed{\int_{-\infty}^a f(x)\,dx = \lim_{R\to-\infty} \int_R^a f(x)\,dx}$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{a} f(x)\,dx}\; + \;\underbrace{\int_{a}^{\infty} f(x)\,dx}}$$

both need to converge (i.e. be finite).



$$\int_{-\infty}^{\infty} f(x)\,dx =$$

$$= \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{-\infty} + \underbrace{\int_{0}^{+\infty} f(x)\,dx}_{\infty} =$$

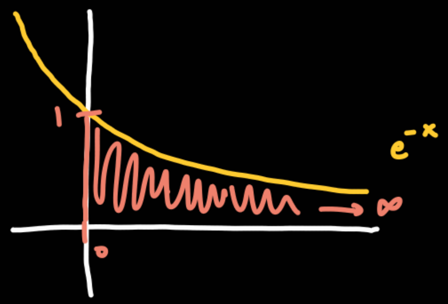
$= -\infty + \infty$, not defined.

$= 0.$

No!

Example: Compute:

$$\int_0^\infty e^{-x}\,dx = \lim_{R\to\infty} \int_0^R e^{-x}\,dx = \lim_{R\to\infty} \left(-e^{-x}\right)\Big|_0^R = \lim_{R\to\infty} \left(-e^{-R} - (-e^{-0})\right) =$$



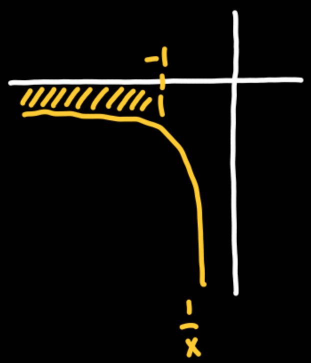
$$= \lim_{R\to\infty} \left(1 - e^{-R}\right) = 1.$$

Example: Compute:

$$\int_2^\infty \frac{1}{x^3}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x^3}\,dx = \lim_{R\to\infty} \left(\frac{-x^{-2}}{2}\right)\Big|_2^R = \lim_{R\to\infty} \left(\frac{-R^{-2}}{2} + \frac{2^{-2}}{2}\right) =$$

$$= \lim_{R\to\infty} \left(\frac{1}{8} - \frac{1}{2R^2}\right) = \frac{1}{8}.$$

$$\int_{-\infty}^{-1} \frac{1}{x}\,dx = \lim_{R\to-\infty} \int_R^{-1} \frac{1}{x}\,dx = \lim_{R\to-\infty} \ln|x|\Big|_R^{-1} = \lim_{R\to-\infty} \left(\ln(1) - \ln|R|\right) =$$

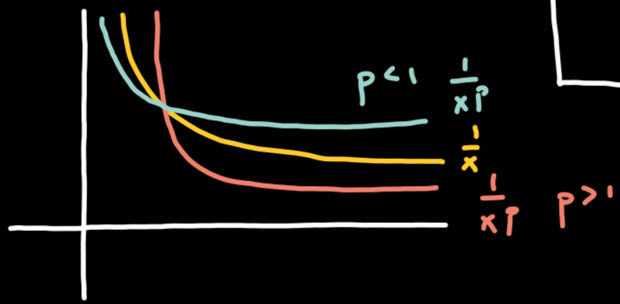


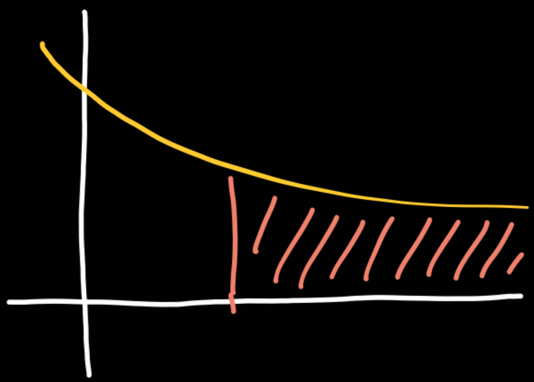
$= -\infty$.

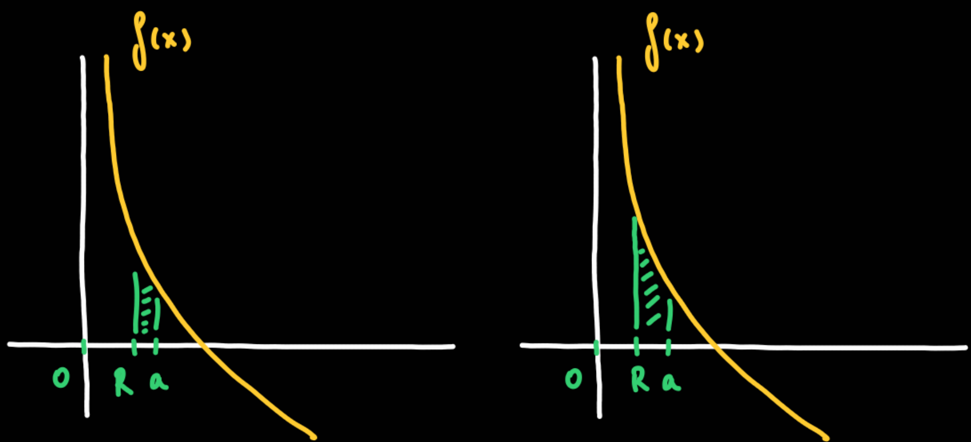
The integral diverges.

$$\boxed{\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{a^{1-p}}{p-1} & p>1. \\ \text{diverge} & p\le 1. \end{cases}}$$

The p-integrals between $[a,\infty)$:







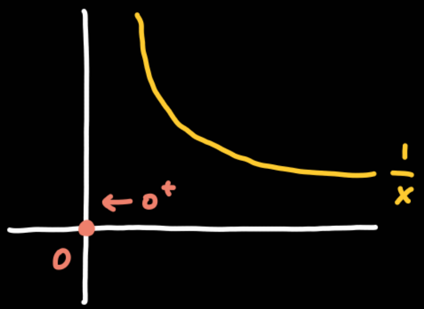
$$\boxed{\int^b f(x)\,dx = \lim_{R\to a^+} \int_R^b f(x)\,dx}$$

$$\boxed{\int_a^b f(x)\,dx = \lim_{R\to b^-}\int_a^R f(x)\,dx}$$

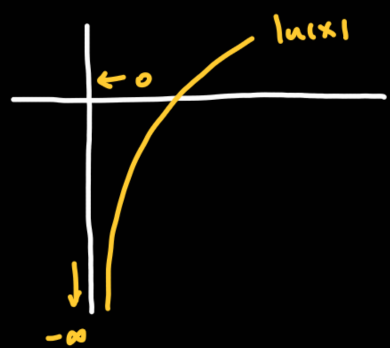
Example: Compute:

$$\int_0^9 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0^+}\int_R^9 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0^+} \left(2x^{1/2}\right)\Big|_R^9 = \lim_{R\to 0^+}\left(6-2\sqrt{R}\right) = 6.$$

$$\int_0^{\frac12} \frac{1}{x}\,dx = \lim_{R\to 0^+}\int_R^{\frac12} \frac{1}{x}\,dx = \lim_{R\to 0^+}\left(\ln|x|\right)\Big|_R^{\frac12} = \lim_{R\to 0^+}\left(\ln\left(\tfrac12\right) - \ln(R)\right) =$$



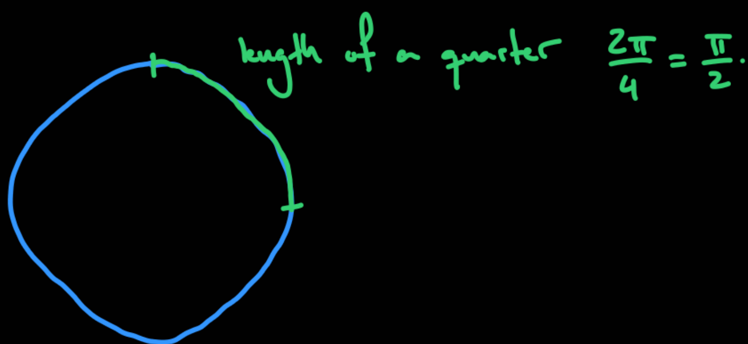
$$= \ln\left(\tfrac12\right) - (-\infty) = +\infty.$$

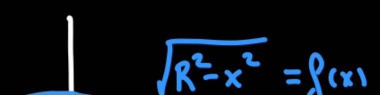


The p-integral over $[0,a]$:

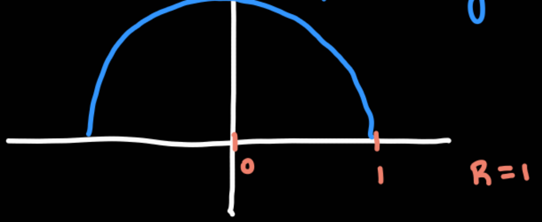
$$\boxed{\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \text{diverges} & p \geq 1. \\ \dfrac{a^{1-p}}{1-p} & p < 1. \end{cases}}$$

Example: Compute the length of one quarter of the unit circle.





arc-length: $\int^b \sqrt{1+(f'(x))^2}\,dx.$



$$f'(x) = \frac{-x}{\sqrt{1-x^2}} \qquad 1+(f'(x))^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}.$$

$$\int_0^1 \sqrt{1+(f'(x))^2}\cdot dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{R\to 1^-} \int_0^R \frac{dx}{\sqrt{1-x^2}} =$$

$$= \lim_{R\to 1^-} \arcsin(x)\Big|_0^R = \lim_{R\to 1^-} \left(\arcsin(R) - \arcsin(0)\right) =$$

$$= \arcsin(1) - \arcsin(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$