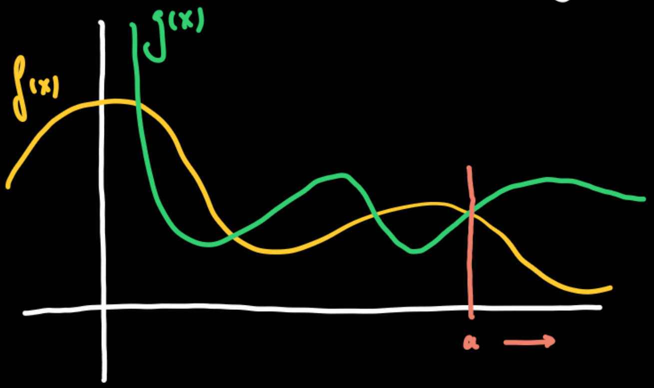
Recall:

$\int_a^\infty \frac{1}{x^p}\,dx$

$\int_0^b \frac{1}{x^p}\,dx$

Comparison test: (for improper integrals)



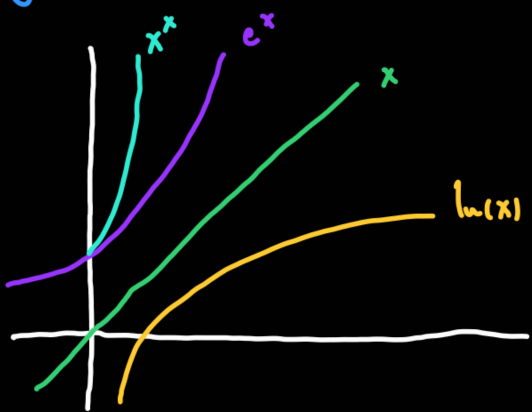
$g(x) \geq f(x)$ for $x > a$.

If $\int_a^\infty g(x)$ converges then $\int_a^\infty f(x)$ converges.

If $\int_a^\infty f(x)$ diverges then $\int_a^\infty g(x)$ diverges.

Question: Order in increasing strength: $\ln(x)$, x^n, e^x, x^x.

($n!$) When $x \to \infty$, which of these functions is the largest (i.e. grows the fastest?



$x^x \gg e^x \gg x^n \gg \ln(x)$ fixed n.

If we allow n to change then

$n!$ is also very strong.

$x!$

Examples: Does the integral $\int_1^\infty \frac{e^{-x}}{x}\,dx$ diverge or converge?

$\frac{e^{-x}}{x}$ can be compared with $\frac{1}{x}$ or e^{-x}.

If we think that the original integral diverges, we want to compare with $\frac{1}{x}$.

$\frac{1}{x} \overset{?}{\leq} \frac{e^{-x}}{x}$ $\quad x=1 \rightsquigarrow 1 \overset{?}{\leq} e^{-1} = \frac{1}{e}$ $\quad$ NOT true.

If we think that the original integral converges, we want to compare with e^{-x}

$\frac{e^{-x}}{x} \overset{?}{\leq} e^{-x}$ $\quad x=1 \rightsquigarrow \frac{1}{e} = e^{-1} \overset{?}{\leq} 1$ $\quad$ True!

$$\int_1^\infty \frac{e^{-x}}{x}\,dx = \underbrace{\int_1^\infty \frac{1}{e^x}\cdot\frac{1}{x}\cdot dx}$$

$$e^x, \quad x$$

$$\int_1^\infty \frac{1}{e^x}\,dx, \quad \int_1^\infty \frac{1}{x}\,dx$$

The inequality $\underline{\frac{e^{-x}}{x}} \leq e^{-x}$ is true in the interval $[1,\infty)$.

Dividing by a number bigger than 1 decreases the original number.

By the comparison test:

$$\int_1^\infty \frac{e^{-x}}{x}\,dx \leq \int_1^\infty e^{-x}\,dx = \frac{1}{e}$$

done in the past

. So the integral converges.

Also: $\frac{e^{-x}}{x} \leq \frac{1}{x}$. Then:

$$\int_1^\infty \frac{e^{-x}}{x}\,dx \leq \int_1^\infty \frac{1}{x}\,dx = \infty$$

(diverges). The Comparison test does not give any information.

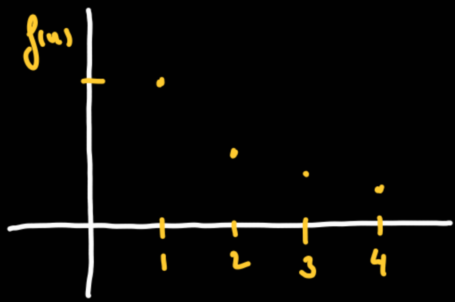
Section 11.1: Sequences:

A sequence is an ordered list of numbers.

$f(n) = \underbrace{\frac{1}{n}}_{\text{general term}}$ $\qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

$\qquad\qquad\qquad\qquad 1 \quad 2 \quad 3 \quad 4 \quad \dots$

a_n



$f(n) = a_n$ defined recursively:

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right), \quad a_1 = 1$$

$$a_1 = 1$$

$$a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}(1+2) = \frac{3}{2}$$

$$a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3}\cdot 2\right) = \frac{17}{12}.$$

$$a_4 = \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{12}{17}\cdot 2\right) = \dots$$

Interpolation:

$1, 3, 68, 71, ?$

N

There is a polynomial $f(x)$ such that $f(1) = 1$, $f(2) = 3$, $f(3) = 68$, $f(4) = 71$, $f(5) = N$.

Fibonacci sequence: $\quad a_1 = 1, \; a_2 = 1, \; a_n = a_{n-1} + a_{n-2}$

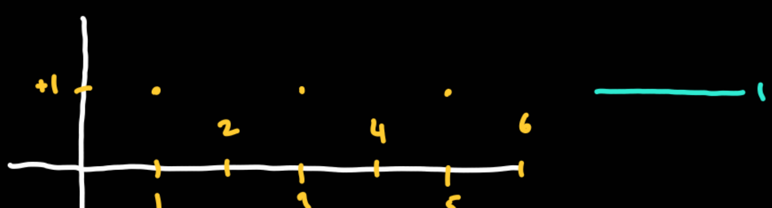
$$1, 1, 2, 3, 5, 8, \dots$$

Limit of a sequence:

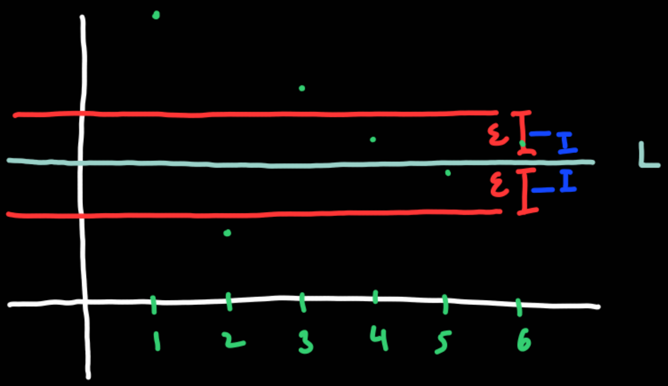
Does my list of numbers get arbitrarily close to a specific number L?

from somewhere onwards

we stay arbitrarily close.



A sequence $\{a_n\}_{n=1}^{\infty}$ has limit L if for every $\varepsilon > 0$ there is some integer M such that $|a_n - L| < \varepsilon$ for all $n > M$.



<u>Limit of a sequence given by a function:</u>

$f(n) = a_n$ and $\lim\limits_{x\to\infty} f(x)$ converges then $\underbrace{\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x)}$.

the limit of the sequence is the limit of the function.

<u>Example:</u> Compute $\lim\limits_{n\to\infty} \dfrac{n+\ln(n)}{n^2}$.

$f(x) = \dfrac{x+\ln(x)}{x^2}$. ← function.

If $\lim\limits_{x\to\infty} f(x)$ converges then:

$$\lim_{n\to\infty} \frac{n+\ln(n)}{n^2} = \lim_{x\to\infty} \frac{x+\ln(x)}{x^2} = \frac{\infty}{\infty} \overset{\text{LHR}}{=} \lim_{x\to\infty} \frac{1+\frac{1}{x}}{2x} = 0.$$

x^2 is stronger than $\ln(x)$

$$\underbrace{\frac{1}{x}}_{0} + \underbrace{\frac{\ln(x)}{x^2}}_{0}$$

Example: For $r \geq 0$ and $c > 0$, compute:

$$\lim_{n\to\infty} c \cdot r^n = \lim_{x\to\infty} c \cdot r^x = c \cdot \lim_{x\to\infty} r^x = \begin{cases} 0 & \text{if } 0 \leq r < 1 \\ c & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

constant   rate/radius

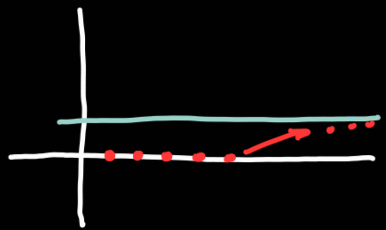
Whenever sequences converge, they can be treated as numbers:

if $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$ then:

(1) $\lim\limits_{n\to\infty} (a_n + b_n) = L + M$

(2) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = L \cdot M$

(3) $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{M}$ if $M \neq 0$.



(4) $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot L$

Squeeze theorem: Given sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ such that $a_n \leq b_n \leq c_n$ for $n > M$, M some integer.

and $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, then $\lim\limits_{n\to\infty} b_n = L$.