Recall: $\lim_{n\to\infty} a_n = L \qquad \lim_{n\to\infty} b_n = M$

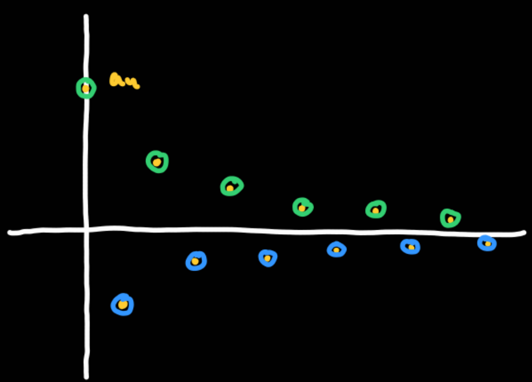
(i) We can add: $\{a_n + b_n\}$ has limit $L+M$.

(ii) We can multiply: $\{a_n \cdot b_n\}$ $\qquad L \cdot M$.

(iii) We can divide: $\left\{\frac{a_n}{b_n}\right\}$ $\qquad \frac{L}{M}$ when $M \neq 0$.

(iv) We can multiply by constants: $\{c \cdot a_n\}$ $\qquad c \cdot L$.

Squeeze theorem: a sequence sandwiched between two sequences with the same limit will also have that limit.



If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$ because of the Squeeze theorem.

Recall: $$\lim_{n\to\infty} c \cdot r^n = \begin{cases} 0 & \text{if } 0 \leq r < 1. \\ c & \text{if } r = 1. \\ \infty & \text{if } r > 1. \end{cases}$$

Example: Compute for $r < 0$ and $c \neq 0$:

$$\lim_{n\to\infty} c \cdot r^n = \begin{cases} 0 & \text{if } -1 < r \leq 0 \\ \text{diverges} & \text{if } r \leq -1 \end{cases}$$

$r = -1$

Example: Compute lim R^n for all R

Example: Compute $\lim_{n\to\infty} \frac{R^n}{n!}$ for all R.

R^n exponential $\qquad\qquad \lim_{n\to\infty} \frac{R^n}{n!} = 0.$

$n!$ factorial $\longleftarrow$ is stronger when $n \to \infty$.

$R>0$: We know that $n! < R^n$ for $n < R$. Let M be the positive integer such that $M \le R < M+1$.

Now for $n > M$:

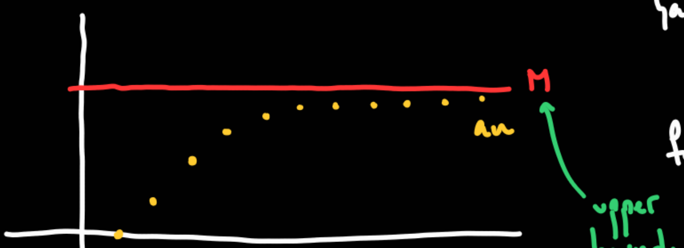
$$\underbrace{\frac{R^n}{n!}}_{\text{positive}} = \underbrace{\left(\frac{R}{1}\cdot\frac{R}{2}\cdot\frac{R}{3}\cdots\frac{R}{M}\right)}_{c} \cdot \underbrace{\frac{R}{M+1}}_{\le 1} \cdot \underbrace{\frac{R}{M+2}}_{\le 1} \cdots \underbrace{\frac{R}{n-1}}_{\le 1} \cdot \frac{R}{n} \le c \cdot \underbrace{\frac{R}{n}}_{\xrightarrow[n\to\infty]{} 0}$$

So by the Squeeze theorem $\lim_{n\to\infty} \frac{R^n}{n!} = 0.$

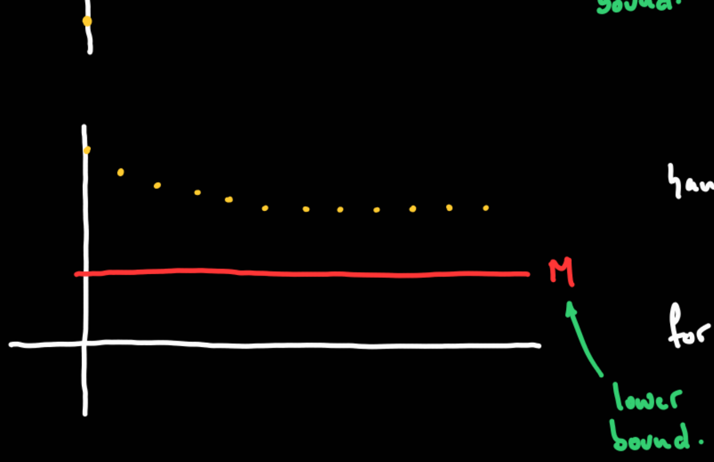
$R<0$: Since $\lim_{n\to\infty} \left|\frac{R^n}{n!}\right| = \lim_{n\to\infty} \frac{|R|^n}{n!} = 0$ then $\lim_{n\to\infty} \frac{R^n}{n!} = 0.$

$$\boxed{\begin{array}{l}\text{If } f(x) \text{ is continuous and } \lim_{n\to\infty} a_n = L \text{ then:} \\ \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).\end{array}}$$

Bounded:

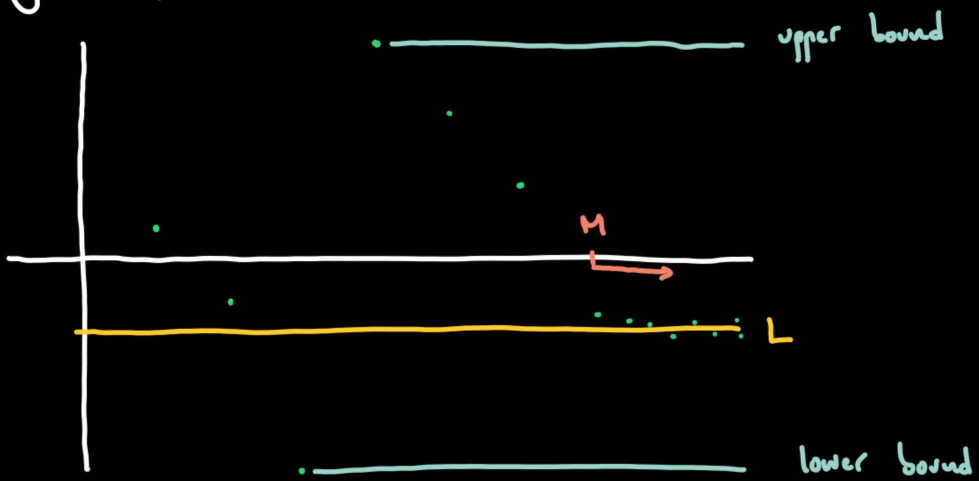


$\{a_n\}$ is bounded above by M if $a_n \le M$ for all n.



$\{a_n\}$ is bounded below by M if $a_n \geq M$ for all n.

Convergent sequences are always bounded.



Bounded sequences that are monotonic converge.

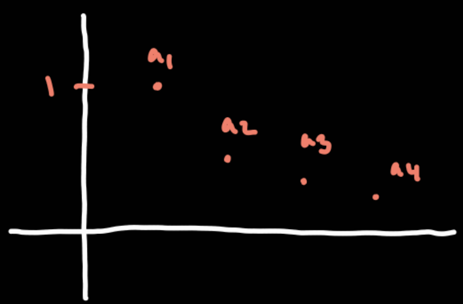
① Monotonically increasing and bounded above implies convergent.

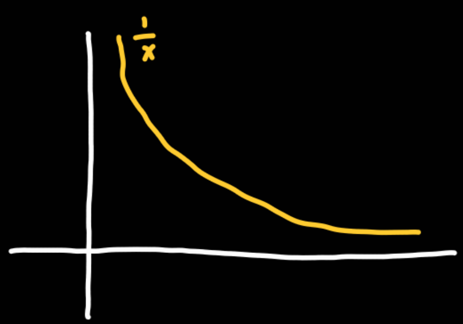
② Monotonically decreasing and bounded below implies convergent.

<u>Example:</u> Is $\frac{1}{n}$ increasing or decreasing or neither?

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

How about $\frac{1}{x}$?





COMPUTE THE DERIVATIVE!

Example: Does the $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ exist?

1. This is decreasing: $f(x)=\sqrt{x+1}-\sqrt{x}$

$$f'(x)=\frac{1}{2\sqrt{x+1}}-\frac{1}{2\sqrt{x}}<0 \quad \text{for } x>0$$

$$\frac{1}{2\sqrt{x+1}}\overset{?}{<}\frac{1}{2\sqrt{x}} \qquad \sqrt{x}\overset{?}{<}\sqrt{x+1} \qquad \text{Yes!}$$

2. Also $\sqrt{n+1}-\sqrt{n}$ is always bigger than 0, so bounded below.

So it has a limit.

To do: Compute the limit by multiplying by $\dfrac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

Section 11.2.: Summing an infinite series.

Sequence: list of numbers. $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, \dots$

add the sequence op.

Series: infinite sum. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots=\frac{\pi}{4}$.

How to compute the sum of an infinite series: $\sum_{n=1}^{\infty} a_n$?

We do the same thing we did for integrals:

$$\int_a^\infty f(x)\,dx=\lim_{R\to\infty}\int_a^R f(x)\,dx \rightsquigarrow \boxed{\sum_{n=1}^{\infty} a_n=\lim_{N\to\infty}\underbrace{\sum_{n=1}^{N} a_n}}$$

$S_N=\sum_{n=1}^{N} a_n$ are the partial sums.

$S_1, S_2, S_3, \dots, S_N, \dots$

sequence of partial sums.

If $\lim_{N\to\infty} S_N$ is finite then the infinite series converges. Otherwise the infinite series diverges.