Recall: A sequence is a list: $\{a_n\}_{n=1}^{\infty}$.

$$1, -1, 1, -1, 1, -1, \ldots$$

An infinite series is the sum of a sequence: $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{} = \lim_{N\to\infty} \underbrace{S_N}_{}$$

partial sums

$$\underset{S_1}{1},\ \underset{S_2}{0},\ \underset{S_3}{1},\ \underset{S_4}{0},\ \underset{S_5}{1},\ \underset{S_6}{0}$$

For $\sum_{n=1}^{\infty} a_n$ to be finite (i.e. to converge) we must have $\lim_{n\to\infty} a_n = 0$.

Telescopic series: Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

partial fraction decomposition

$$a_n = \frac{1}{n(n+1)} \qquad S_N = \sum_{n=1}^{N} a_n \qquad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$a_1 = \frac{1}{1\cdot 2} = \frac{1}{2} \qquad S_1 = a_1 = \frac{1}{2} \qquad S_1 = 1 - \frac{1}{2}$$

$$a_2 = \frac{1}{2\cdot 3} = \frac{1}{6} \qquad S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} \qquad S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$a_3 = \frac{1}{3\cdot 4} = \frac{1}{12} \qquad S_3 = a_1 + a_2 + a_3 = \cdots = \frac{9}{12}. \qquad S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$a_4 = \frac{1}{4\cdot 5} = \frac{1}{20} \qquad S_4 = \cdots = \cdots \qquad \vdots$$

$$\vdots$$

$$S_N = 1 - \frac{1}{N+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = \underbrace{\left(\lim_{N\to\infty} 1\right)}_{1} - \underbrace{\left(\lim_{N\to\infty} \frac{1}{N+1}\right)}_{0} = 1$$

Aside: If $\sum a_n$ and $\sum b_n$ converge, then:

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$$\sum (a_n + b_n) = \sum a_n + \sum b_n \text{ is convergent.}$$

$$\sum (a_n - b_n) = \sum a_n - \sum b_n \text{ is convergent.}$$

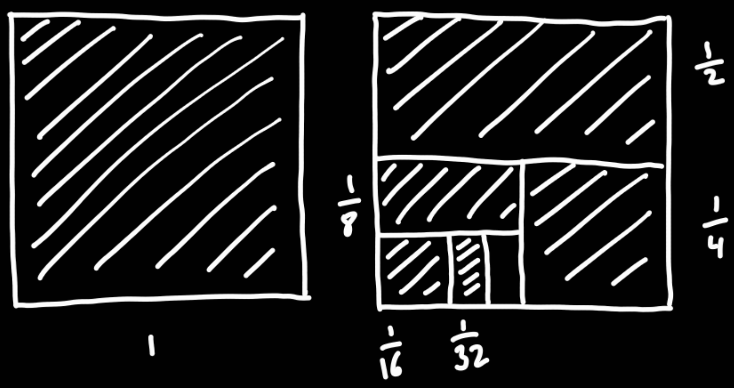
$$\sum c \cdot a_n = c \cdot \sum a_n \text{ is convergent for all real number } c.$$

Careful! Nothing is said about $\sum \frac{a_n}{b_n}$!

$a_n = (-1)^n \cdot \frac{1}{n^2}$, $b_n = (-1)^{n+1} \cdot \frac{1}{n^2}$, $\frac{a_n}{b_n}$ can have alternating signs!

Geometric series: Compute $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

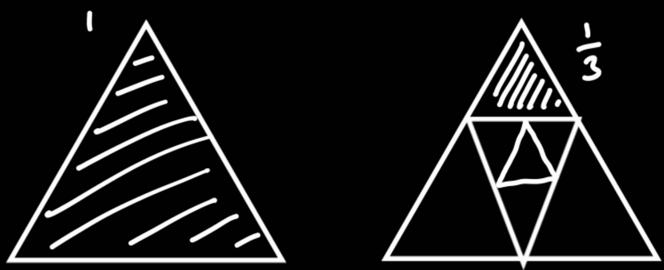
$a_n = \frac{1}{2^n}$ $\quad 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \cdots$



Intuitively:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

Compute $\sum_{n=0}^{\infty} \frac{1}{3^n}$.



In general: $\boxed{\sum_{n=0}^{\infty} c \cdot r^n = c + c \cdot r + c \cdot r^2 + \cdots = \frac{c}{1-r}.}$ for $|r| < 1$.

$c = 1 \quad r = \frac{1}{2}$ then $\frac{c}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{2-1}{2}} = \frac{2}{1} = 2.$

$$\sum_{n=M}^{\infty} c \cdot r^n = c \cdot r^M + c \cdot r^{M+1} + c \cdot r^{M+2} + \cdots = r^M \cdot \frac{c}{1-r}.$$

Compute:

$$\sum_{n=0}^{\infty} \frac{2+3^n}{5^n} = \sum_{n=0}^{\infty} \frac{2}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^n} = 2 \cdot \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n =$$

$$= 2 \cdot \frac{1}{1-\frac{1}{5}} + \frac{1}{1-\frac{3}{5}} = 2 \cdot \frac{1}{\frac{4}{5}} + \frac{1}{\frac{2}{5}} = \frac{5}{2} + \frac{5}{2} = 5.$$

Today:

(1) For a series to converge we need the general term to have limit zero.

(2) Telescopic series.

(3) Geometric series.