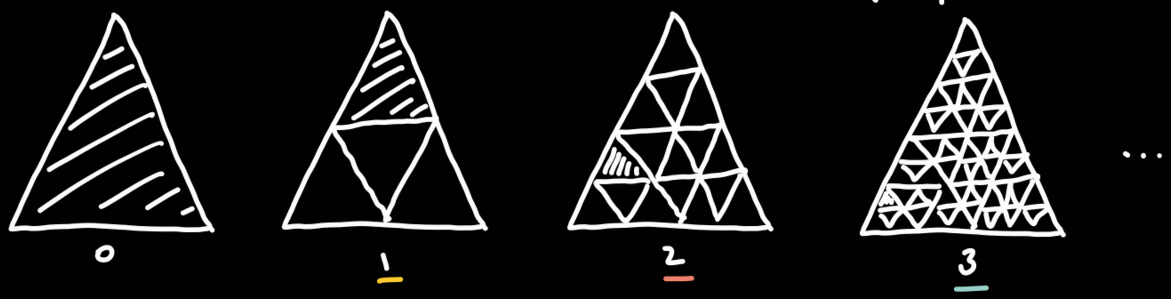
__Recall:__ Sequences: fancy lists of numbers.

Infinite series: $\sum_{n=1}^{\infty} a_n$. $\quad \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{S_N \text{ partial sums}}$

$S_1, S_2, S_3, S_4, \dots$

Geometric series: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = \text{magic picture} = \frac{1}{3} \qquad \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{4}{4}-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$



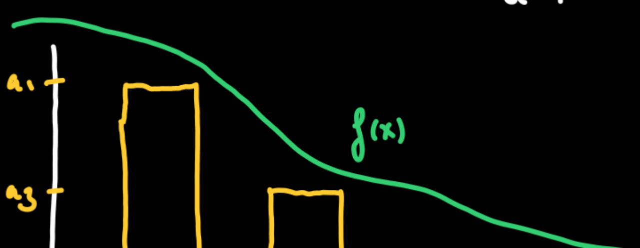
$$\frac{4}{3} = \frac{3}{3} + \frac{1}{3} = \underbrace{1}_{n=0} + \underbrace{\frac{1}{3}}_{\text{sum of } n>0}.$$

__Section 11.3.:__ Series with positive terms.

We want to use integration techniques to determine convergence and divergence of infinite series.

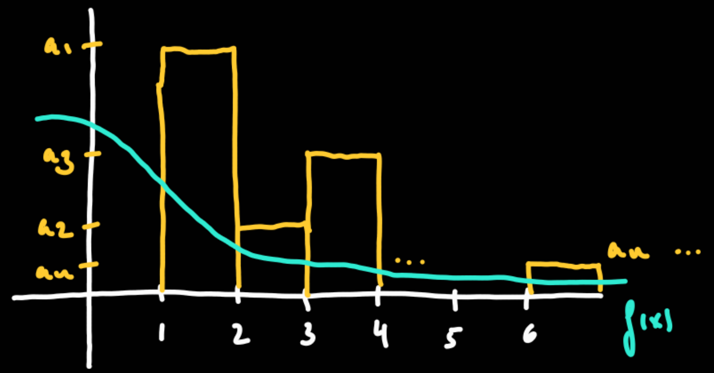
An integral can be interpreted as an area.

We first have to interpret $\sum_{n=1}^{\infty} a_n$ as an area ($a_n \geq 0$).



$$\int_1^{\infty} f(x)\,dx \geq \sum_{n=1}^{\infty} a_n$$

$f(n) = a_n$ and other conditions



$$\int_1^\infty f(x)\,dx \le \sum_{n=1}^{\infty} a_n$$

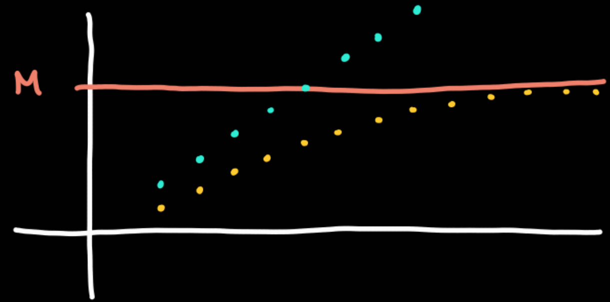
$f(n) \le a_n$ and more ...

<u>Positive series:</u> $\sum_{n=1}^{\infty} a_n$ with $a_n > 0$.

$S_N < S_{N+1}$, partial sums are increasing.

(i) Partial sums bounded: convergence of the infinite series.

(ii) Partial sums unbounded: divergence of the infinite series.



<u>Integral test:</u> If $f(x)$ is decreasing and $f(n) = a_n$ (and continuous) then:

(i) $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) $\int_1^\infty f(x)\,dx$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Example:</u> Harmonic series. $\sum_{n=1}^{\infty} \frac{1}{n}$ $f(x) = \frac{1}{x}$

0th harmonic.

$\int^\infty \ldots = \lim \int^R \ldots = \lim \ln(R) = \ldots$

$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{R\to\infty}\int_1^{R} \frac{1}{x}\,dx = \lim_{R\to\infty} \ln(R) = \infty.$

1st harmonic.

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

2nd harmonic.

$\vdots$

The general term $\frac{1}{n}$ has limit zero but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Namely, if an infinite series $\sum_{n=1}^{\infty} a_n$ diverges then it is not true that $\lim_{n\to\infty} a_n \neq 0$.

Example: Determine whether $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges or diverges.

these must be positive ①

We are checking the hypothesis of the Integral Test.

② always choose this

$f(n) = a_n$

$f(x) = \frac{x}{(x^2+1)^2}$ is positive for $x \geq 1$.

we do not have to check this.

To check that it is decreasing we compute the derivative:

$$f'(x) = \frac{1-3x^2}{(x^2+1)^3} < 0 \text{ for } x \geq 1, \text{ so } f(x) \text{ is decreasing. ③}$$

CHECK THE DETAILS!

$$\int_1^{\infty} \frac{x}{(x^2+1)^2}\,dx = \cdots = \frac{1}{4}.$$

Since the integral converges, the infinite series converges.

We cannot guarantee that the sum is $\frac{1}{4}$.

P-series:

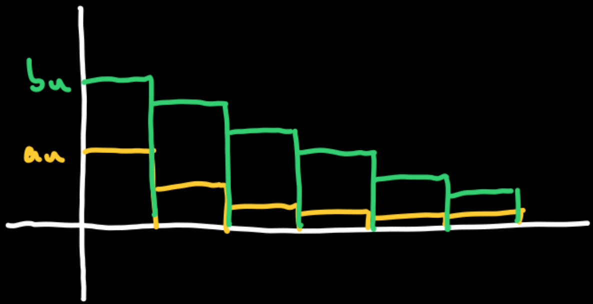
The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge for $p>1$ and diverge for $p \leq 1$.

This is given by applying the Integral Test to $\int_1^{\infty} \frac{1}{x^p}\,dx$.

Comparison Test: If there is an M such that $\underbrace{0 \le a_n \le b_n \text{ for } n > M}_{\text{eventually } a_n \le b_n}$:

(i) If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.



Example: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\cdot 3^n}$.

$\sqrt{n}$

3^n

Compare: $\frac{1}{\sqrt{n}\cdot 3^n} \le \underbrace{\frac{1}{3^n}}_{\text{p-series}}$.

DO THE DETAILS!

(converges).

$\frac{1}{\sqrt{n}\cdot 3^n} \le \frac{1}{\sqrt{n}}$

gives the infinite series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges

$\int_1^{\infty} \frac{1}{x^{1/2}}\,dx$ diverges. Integral Test.

Example: Determine the convergence or divergence of $\sum_{n=2}^{\infty} \frac{1}{(n^2+3)^{1/3}}$.

$(n^2+3)^{1/3}$ looks like $n^{2/3}$: $\underbrace{\sum_{n=2}^{\infty} \frac{1}{n^{2/3}}}_{\text{divergent p-series.}}$

So we want to use the Comparison Test to see that $\sum_{n=2}^{\infty} \frac{1}{(n^2+3)^{1/3}}$ diverges.

$\frac{1}{n^{1/3}} \le \frac{1}{(n^2+3)^{1/3}}$ is <u>not</u> true for $n \ge 2$

$$\frac{1}{n} \le \frac{1}{n^2+3} \rightsquigarrow n^2+3 \le n \quad \text{for } n \ge 2$$

We would like to compare:

$\frac{1}{n^{2/3}} \le \frac{1}{(n^2+3)^{1/3}}$ is <u>not</u> true for $n \ge 2$.

But we know:

$\frac{1}{n} \le \frac{1}{n^{2/3}}$, so we would like $\frac{1}{n} \overset{?}{\le} \frac{1}{(n^2+3)^{1/3}}$.

$$\Updownarrow$$

$$n^2+3 \overset{?}{\le} n^3 \rightsquigarrow (n^2+3)^{1/3} \le (n^3)^{1/3}$$

$$\Updownarrow$$

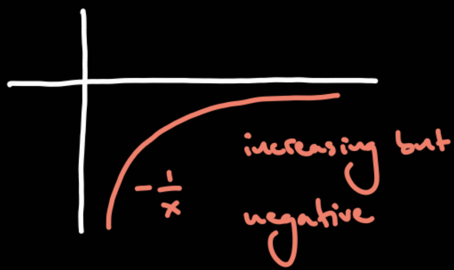
$x^3-(x^2+3) \ge 0$ $\quad$ $f(x) = x^3-(x^2+3)$ should be positive.

$x \ge 2$

Now:

$f'(x) = 3x^2-2x = 3x(x-\frac{2}{3}) > 0$ for $x \ge 2$. So $f(x)$ is increasing.

$f(2) = 8-7 = 1$.



So $f(x)$ is positive for $x \ge 2$.

By the comparison test, since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges then $\sum_{n=2}^{\infty} \frac{1}{(n^2+3)^{1/3}}$ diverges.