Recall: We were determining convergence and divergence of series with positive terms.

Integral test.

Comparison test.

Limit comparison test: $\sum a_n$, $\sum b_n$ positive series such that $L = \lim_{n\to\infty} \frac{a_n}{b_n}$

exists.

a_n smaller than b_n — If $L=0$ and $\sum b_n$ converges then $\sum a_n$ also converges.

b_n smaller than a_n — If $L=\infty$ and $\sum a_n$ converges then $\sum b_n$ also converges.

a_n and b_n are equivalent — If $L>0$ real number, $\sum a_n$ converges if and only if $\sum b_n$ converges.

① $\frac{a_n}{b_n} = (-1)^n$, $L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} (-1)^n$, which does <u>not</u> exist.

(also the series are <u>not</u> positive)

② a_n : 2, 1, 2, 1, 2, 1, ...

(neither has finite sum)

b_n : 1, 1, 1, 1, 1, 1, ...

$\frac{a_n}{b_n}$: 2, 1, 2, 1, 2, 1, ... $\leftarrow$ does <u>not</u> have a limit.

③ a_n : $\frac{1}{n^2}$ $\leftarrow$ $\sum a_n$ converges (p-series).

b_n : $\begin{cases} \frac{1}{n^2} & n \text{ even} \\ \frac{1}{n^3} & n \text{ odd} \end{cases}$ $\leftarrow$ $\sum b_n$ converges (Comparison Test with p-series).

$\frac{a_n}{b_n}$: $\begin{cases} 1 & n \text{ even} \\ n & n \text{ odd} \end{cases}$ $\leftarrow$ $\lim_{n\to\infty} \frac{a_n}{b_n}$ has <u>no</u> limit.

two positive convergent series where the Limit Comparison Test does not apply.

Example: Determine convergence or divergence of: $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$.

How does $\underbrace{\frac{n^2}{n^4-n-1}}$ behave for really big n?

$\frac{n^2}{n^4}$, i.e. $\frac{1}{n^2}$, a converging p-series.

However: $\frac{1}{n^2} \leq \frac{n^2}{n^4-n-1}$, so we can't apply the Comparison Test.

$a_n = \frac{1}{n^2}$, $b_n = \frac{n^2}{n^4-n-1}$, now: $L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4-n-1}{n^4} = 1$.

By the Limit Comparison Test, since $\sum a_n$ converges, then $\sum b_n$ converges.

Section 11.4.: Absolute and conditional convergence. (series are no longer positive series)

A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

A series $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Q: If $\sum a_n$ is absolutely convergent, why do we not require $\sum a_n$ to converge?

A: If $\sum |a_n|$ converges then $\sum a_n$ converges.

If $\sum a_n$ converges absolutely then $\sum a_n$ converges.

Example: Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$, converging p-series. So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely.

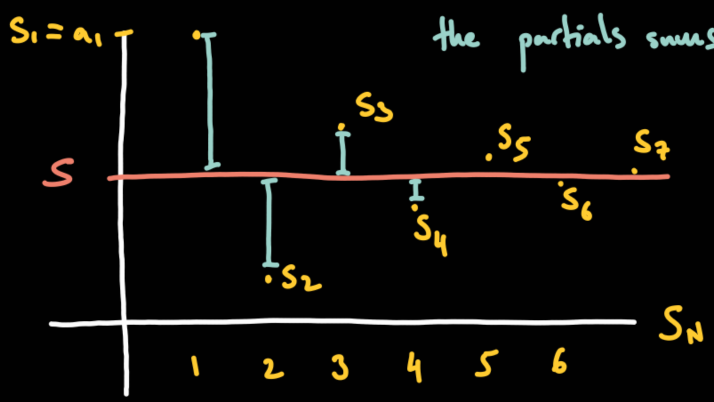
Careful: $\sum |a_n| \neq \left|\sum a_n\right|$.

__Leibniz test:__ (for alternating series: we add positive and negative numbers in an alternating fashion)

$\{a_n\}$ positive sequence, decreasing, $\lim_{n\to\infty} a_n = 0$. ← DO NOT FORGET TO CHECK THE HYPOTHESES

Then $S = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n$ converges, $0 < S < a_1$, and $S_{2N} < S < S_{2N+1}$

for all natural numbers N.



the partials sums sandwich S.

__Example:__ Alternating harmonic series: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

Determine the type of convergence of this series (if any).

1) $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

So the alternating harmonic series is __not__ absolutely convergent.

2) Consider $a_n = \frac{1}{n}$, it is decreasing, and $\lim_{n\to\infty} \frac{1}{n} = 0$.

By the Leibniz test, the alternating harmonic series converges.

So the alternating harmonic series converges conditionally.

If $\{a_n\}$ is positive, decreasing, and $\lim_{n \to \infty} a_n = 0$, set $S = \sum_{n=1} (-1)^{n-1} \cdot a_n$, then:

$$|S - S_N| < a_{N+1}.$$

$$S_N = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{N-1} \cdot a_N$$

In other words, the error that we make by approximating an alternating sum by the N-th partial sum is less than the $(N+1)$-th term.