Recall:

	$\sum a_n$	$\sum \lvert a_n \rvert$
Absolute convergence	converges	converges
Conditional convergence	converges	diverges

Section 11.5.: The ratio and root tests.

Ratio test: If $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then:

(i) if $\rho < 1$ then $\sum a_n$ converges absolutely.

(ii) if $\rho > 1$ then $\sum a_n$ diverges.

(iii) if $\rho = 1$ then the test is inconclusive.

Example: Determine convergence (conditional or absolute) or divergence of $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

Weak $\qquad\qquad$ Strong

$\ln(x),\ x^n,\ e^x,\ n!,\ x^x$

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0.$$

$(n+1)! = (n+1)\cdot n!$

$\rho = 0 < 1$ so $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges absolutely.

Example: Determine convergence (conditional or absolute) or divergence of $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{n!}{(n+1)!} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{n^2+1+2n}{n^2} \cdot \frac{1}{n+1} \right| = 0$$

We have absolute convergence by the ratio test.

__Example:__ Determine convergence (conditional or absolute) or divergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| = \lim_{n\to\infty} \left| \frac{n^2}{n^2+1+2n} \right| = 1.$$

$\sum_{n=1}^{\infty} n^2.$

The ratio test is inconclusive.

__Root test:__ If $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then:

x^x

(i) if $L<1$ then $\sum a_n$ converges absolutely.

(ii) if $L>1$ then $\sum a_n$ diverges.

(iii) if $L=1$ then the test is inconclusive.

__Example:__ Determine convergence or divergence of $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$.

For n big then $\frac{n}{2n+3}$ looks like $\frac{n}{2n} = \frac{1}{2}$.

$\left(\frac{n}{2n+3}\right)^n$ looks like $\frac{1}{2^n}$.

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{2n+3}\right)^n\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+3}\right)^n} =$$

$$= \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}.$$

$L = \frac{1}{2} < 1$ so $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ converges absolutely by the root test.

Section 11.6.: Power series.

A power series is an infinite sum with a variable.

$$F(x) = \sum_{n=0}^{\infty} a_n \cdot (x-c)^n = a_0 + a_1 \cdot (x-c) + a_2 \cdot (x-c)^2 + \cdots$$

variable

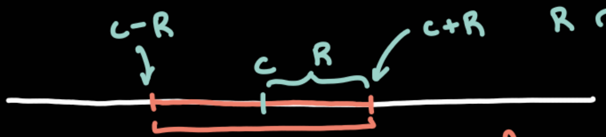
center, $F(c) = a_0$.

real number

$\{a_n\}$ sequence

A power series is an infinite polynomial.

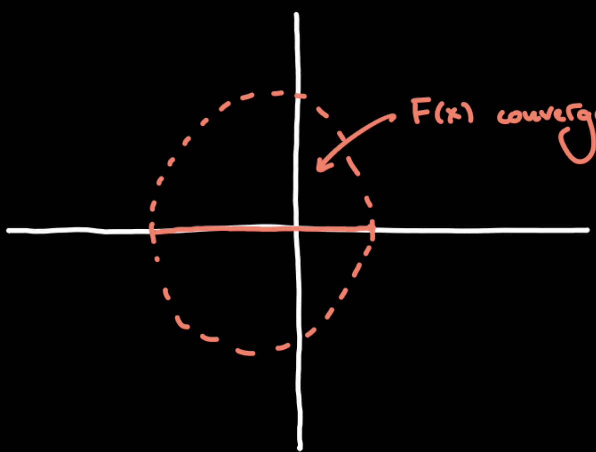
Which are the values of x such that $F(x)$ converges?



R radius of convergence

$\mathbb{R}$

for these values of x we have that $F(x)$ converges.



$F(x)$ converges in this circle (if x is a complex number).

<u>Radius of convergence:</u> A power series $F(x) = \sum_{n=0}^{\infty} a_n \cdot (x-c)^n$ has a radius of convergence R (which can be zero, strictly positive, or infinity).

If R is finite then $F(x)$ converges absolutely when $\underbrace{|x-c| < R}_{\text{moving } x \text{ a distance } R \text{ in either direction}}$, $F(x)$ diverges when $|x-c| > R$.

If R is infinite then $F(x)$ converges absolutely for all x.

The interval where $F(x)$ converges is called the interval of convergence.

How to find the interval of convergence:

1. Find the radius of convergence using the ratio test.

2. Check convergence or divergence at the endpoints.

Example: Determine the interval of convergence of $F(x) = \sum_{n=1}^{\infty} \frac{x^n}{2^n}$.

$F(x) = \sum_{n=1}^{\infty} a_n \cdot (x-c)^n$, $a_n = \frac{1}{2^n}$, $c = 0$.

1. $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{2^n}{2^{n+1}} \right| =$

$\sum_{n=1}^{\infty} \underbrace{\frac{x^n}{2^n}}_{\text{general term}}$. $a_n = \frac{x^n}{2^n}$.

R $|x-c| < R$ convergence

$|x-c| > R$ divergence

$|x| < R$

$|x| > R$

$= \lim_{n\to\infty} \frac{|x|}{2} = \frac{|x|}{2}$.

Now $F(x)$ converges when $\rho < 1$ and diverges when $\rho > 1$.

Convergence: $\frac{|x|}{2} = \rho < 1$ namely $|x| < 2$.

Divergence: $\frac{|x|}{2} = \rho > 1$ namely $|x| > 2$.

So $R = 2$ is the radius of convergence.

2. Check endpoints:

$x = 2$: $F(2) = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = 1+1+1+\cdots$ diverges.

$x = -2$: $F(-2) = \sum \frac{(-2)^n}{\ldots} = 1-1+1-1+\cdots$ divergent.

$n=0 \quad 2^n$

So $F(x)$ converges in the interval $(-2,2)$.

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