<u>Recall:</u> power series $\displaystyle\sum_{n=0}^{\infty} a_n\cdot(x-c)^n = F(x)$

coefficients: a_n center: c

Always have a radius of convergence R.

If x is in $(c-R, c+R)$ then $F(x)$ converges.

$\underbrace{|x-c|<R}_{\text{the distance from } x \text{ to } c \text{ is less than } R.}$

How to find R:

1. Use ratio test.

2. Look at the endpoints.

Geometric series $\left(\displaystyle\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \text{ for } |r|<1\right)$ have been useful. Now:

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x|<1.}$$

"The polynomial $1-x$ has inverse $\displaystyle\sum_{n=0}^{\infty} x^n$." $|x|<1$

Has radius of convergence $1=R$.

Geometric series. Within the radius of convergence, this behaves like a polynomial.

<u>Example:</u> Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} 2^n\cdot x^n$.

$$\sum_{n=0}^{\infty} 2^n\cdot x^n = \sum_{n=0}^{\infty} (2\cdot x)^n = \sum_{n=0}^{\infty} y^n = \frac{1}{1-y} = \frac{1}{1-2x} \qquad |x|<\frac{1}{2}.$$

$|y|<1$

$\sum_{n=0} y^n = \frac{1}{1-y}$ $|y|<1$ $|y|<1$

$|2x|<1$

$y = 2x$ $|x|<\frac{1}{2}$

Also for $x = \pm\frac{1}{2}$ then $\sum_{n=0}^{\infty} 2^n x^n$ diverges.

So $R = \frac{1}{2}$ is the radius of convergence.

<u>Example:</u> Find a power series $F(x)$ and a radius of convergence R such that

$$F(x) = \frac{1}{2+x^2} \quad \text{for } \underline{|x|<R}.$$

the center is $c=0$.

Note:

$$\frac{1}{2+x^2} = \frac{1}{2}\cdot\frac{1}{1+\frac{x^2}{2}} = \frac{1}{2}\cdot\left(\frac{1}{1-\left(\frac{-x^2}{2}\right)}\right) \underset{\uparrow}{=} \frac{1}{2}\cdot\frac{1}{1-y} \underset{\uparrow}{=} \frac{1}{2}\cdot\sum_{n=0}^{\infty} y^n =$$

$y = \frac{-x^2}{2}$ $|y|<1$

$$= \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{-x^2}{2}\right)^n = \frac{1}{2}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{2^n} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{2^{n+1}} \quad \text{for } |x|<\sqrt{2}.$$

$|y|<1$ $\left|\frac{-x^2}{2}\right|<1$ $\frac{|x^2|}{2}<1$ $|x|^2<2$ $|x|<\sqrt{2}$

<u>Term by term differentiation and integration:</u>

$F(x) = \sum_{n=0}^{\infty} a_n\cdot(x-c)^n$ power series, radius of convergence $R>0$.

Then $F(x)$ is differentiable:

$F'(x) = \sum_{n=1}^{\infty} n\cdot a_n\cdot(x-c)^{n-1}$ with radius of convergence R.

Then $F(x)$ is integrable:

$\int F(x)\,dx = A + \sum_{n=0}^{\infty}\frac{a_n}{n+1}\cdot(x-c)^{n+1}$ with radius of convergence R.

a constant.

Example: Find a power series $F(x)$ and a radius of convergence R such that

$$F(x) = \frac{1}{(1-x)^2} \text{ for } \underline{|x| < R}.$$

center $c = 0$.

$\frac{1}{1-x}$ has differential $\frac{1}{(1-x)^2}$

For $|x|<1$ we have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, so:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=1}^{\infty} n \cdot x^{n-1}.$$

So for $|x|<1$ then:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=1}^{\infty} n \cdot x^{n-1}!$$

Exercise: Do this for $\arctan(x)$.

Hint 1: Use integration (term-wise).

Hint 2: Use the geometric series.

Hint 3: Use $\frac{1}{1+x^2}$.

Hint 4: $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad |x|<1.$

As long as we are inside the radius of convergence, power series behave like polynomials: we can add them, we can multiply them, we can evaluate them

($F(x)$ is convergent), we can take derivatives, we can take integrals.

Section 11.7.: Taylor series.

Taylor polynomials approximated functions using derivatives.

$f(x)$ $\quad$ $T_n(x)$

Taylor series: approximation using all the derivatives. $\quad$ $f(x)$

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} \cdot (x-c)^n$$

It will have some radius of convergence R, so $T(x)$ will converge for $|x-c|<R$.

This will work best when $f(x)$ can be written as a power series centered at c.

If $f(x)$ can be written as a power series, then $T(x)$ will be that power series.

Question: For all functions $f(x)$ there is a n such that $T_n(x) = f(x)$.

Question: For all functions $f(x)$ there is a power series $F(x)$ such that

$F(x) = f(x)$. $\quad$ Spoiler: $f(x) = e^{-\frac{1}{x^2}}$ around $c=0$.

Compute $T_n(x)$ for all n.

Compute its Taylor series.

Example: Find the Taylor series of $f(x) = \frac{1}{x^3}$ at $c=1$. $\quad$ $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} \cdot (x-c)^n$

$$f'(x) = \frac{-3}{x^4} \qquad f''(x) = \frac{3\cdot 4}{x^5} \qquad f'''(x) = \frac{-3\cdot 4\cdot 5}{x^6}, \ldots, \quad f^{(n)}(x) = \frac{(-1)^n \cdot 3\cdot 4 \cdots (n+2)}{x^{n+3}}$$

$$3\cdot 4\cdot 5 \cdots (n+2) = \frac{(n+2)!}{2} \qquad (n+2)! = (n+2)\cdot(n+1)\cdots 5\cdot 4\cdot 3\cdot 2$$

$$f^{(n)}(1) = (-1)^n\, 3\cdot 4 \cdots (n+2) = (-1)^n \cdot \frac{(n+2)!}{2}$$

$$\frac{f^{(n)}(1)}{n!} = (-1)^n \cdot \frac{(n+2)!}{2} \cdot \frac{1}{n!} = (-1)^n \cdot \frac{(n+2)(n+1)\cdot n!}{2\cdot n!} = (-1)^n \cdot \frac{(n+2)(n+1)}{2}$$

$$(n+2)! = (n+2)\cdot(n+1)\cdot n!$$

So the Taylor series of $f(x) = \frac{1}{x^3}$ around $c=1$ is:

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(n+2)(n+1)}{2} \cdot (x-1)^n .$$

Let $I = (c-R, c+R)$, $\underbrace{R>0}_{\text{radius of convergence}}$, and a $K>0$ such that $|f^{(n)}(x)| \le K$ for all n and $|x-c| < R$. Then $f(x)$ equals its Taylor series for $|x-c|<R$.

$T(x)$ converges and $T(x) = f(x)$ for x in $(c-R, c+R)$.