Recall: Taylor series: given $f(x)$ we produced $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} \cdot (x-c)^n$.

$\frac{f^{(n)}(c)}{n!}$: ensures that the derivatives of $T(x)$ at c coincide with the derivatives of $f(x)$ at c.

c: center

This $T(x)$ approximates $f(x)$.

Question: When is $T(x)$ equal to $f(x)$?

1. If $f(x)$ equals a power series then this power series is the Taylor series.

2. If $|f^{(n)}(x)| \le K$ for all n and all x in $(c-R, c+R)$ then $f(x)$ equals a power series in $(c-R, c+R)$.

c: center

R: radius of convergence.

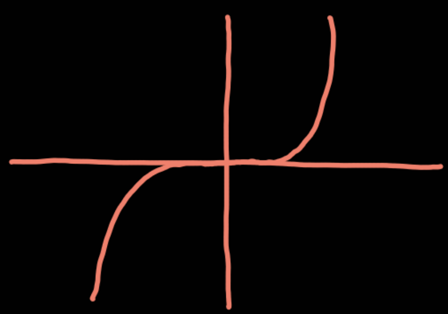
This doesn't always work: $f(x) = e^{-\frac{1}{x^2}}$.

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0. \\ 0 & x = 0. \end{cases}$$

This piecewise defined function is infinitely differentiable.

$f^{(n)}(0) = 0$ for all n.

$T(x) = 0$.



Example: Compute the Maclaurin series of $\sin(x)$.

Maclaurin: Taylor centered at $c=0$.

$f(x) = \sin(x)$ $\quad f(0) = 0$

$f'(x) = \cos(x)$ $\quad f''(x) = -\sin(x)$ $\quad f'''(x) = -\cos(x)$ $\quad f^{iv}(x) = \sin(x)$

$f'(0) = 1$ $\quad f''(0) = 0$ $\quad f'''(0) = -1$ $\quad f^{iv}(0) = 0$

odd 1 even zero odd -1 even zero

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$$

$$a_N = \frac{f^{(N)}(c)}{N!} \qquad |f^{(N)}(x)| \le 1 \text{ for all } N \text{ and all real numbers } x.$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all real numbers } x.$$

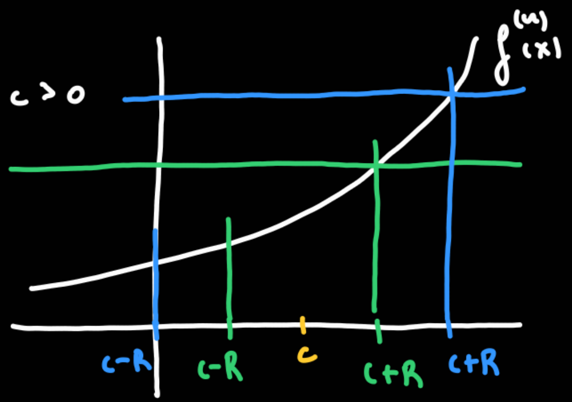
Example: Compute the Maclaurin series of $\cos(x)$.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all real numbers } x.$$

Example: Compute the Taylor series of e^x around c.

$$f(x) = e^x \qquad f^{(n)}(x) = e^x$$

$$f(c) = e^c \qquad f^{(n)}(c) = e^c \qquad a_n = \frac{f^{(n)}(c)}{n!} \qquad a_n = \frac{e^c}{n!}$$

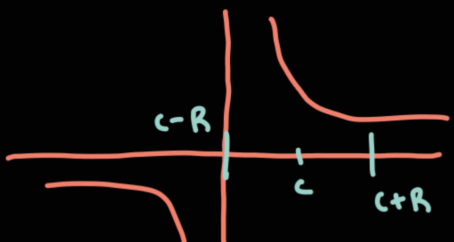


In $(c-R, c+R)$ we have $|f^{(n)}(x)| \le e^{c+R}$.

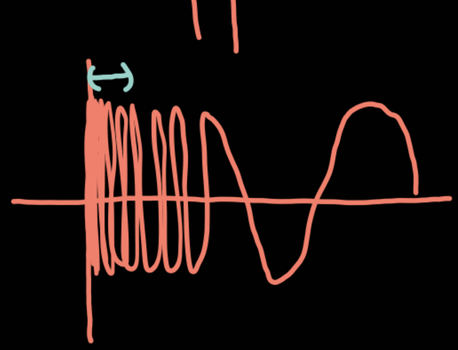
In $(c-R, c+R)$ we have $|f^{(n)}(x)| \le e^{c+R}$.

$$e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n \quad \text{for all real numbers } x.$$

Warning! These ideas do not work for things with discontinuities like $\frac{1}{x}$.



$\sin\left(\frac{1}{x}\right)$



Example: Compute the Maclaurin series of $x^2 \cdot e^x$.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad \text{for all } x.$$

x^2 is its own Maclaurin series for all x.

$$x^2 \cdot e^x = x^2 \cdot \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right) = x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}.$$

The Maclaurin series of $x^2 \cdot e^x$ is $\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$ for all real numbers x.

Example: Compute the Maclaurin series of $\ln(1+x)$.

$\frac{d}{dx}\left(\ln(1+x)\right) = \frac{1}{1+x}$ which looks like a geometric series.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + x^4 - \cdots \quad \text{for } |-x| < 1$$

$|x| < 1.$

Integrating term by term:

$$\ln(1+x) = \int \frac{1}{1+x}\, dx = \int (1 - x + x^2 - x^3 + x^4 - \cdots)\, dx =$$

$$= \int dx - \int x\, dx + \int x^2 dx - \int x^3 dx + \int x^4 dx - \cdots =$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + A$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n + A \quad \text{for } |x| < 1.$$

For $x=0$ we have: $\ln(1+0) = \ln(1) = 0$ $\qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\cdot x^n$

$$\underbrace{A + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\cdot 0}_{0} = 0 \qquad \text{so } A = 0$$

Binomial series: $\boxed{(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n}$ for $|x|<1$ and all a.

$$\binom{a}{n} = \frac{a\cdot(a-1)\cdot(a-2)\cdots(a-n+1)}{n!}$$

Example: Compute the Maclaurin series of $\frac{1}{\sqrt{1-x^2}}$.

First compute the Maclaurin series of $\frac{1}{\sqrt{1+y}}$.

Second substitute $y=-x^2$.

$$\frac{1}{\sqrt{1+y}} = (1+y)^{-\frac{1}{2}} \qquad \binom{a}{n} = \frac{a\cdot(a-1)\cdot(a-2)\cdots(a-n+1)}{n!}$$

n	0	1	2	3
$\binom{a}{n}$	1	$\frac{-\frac{1}{2}}{1!} = \frac{-1}{2}$	$\frac{(\frac{-1}{2})\cdot(\frac{-3}{2})}{2!} = \frac{1\cdot 3}{2\cdot 4}$	$\frac{(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})}{3!} = \frac{-1\cdot 3\cdot 5}{2\cdot 4\cdot 6}$

$$\binom{-\frac{1}{2}}{n} = (-1)^n \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}.$$

$$\frac{1}{\sqrt{1+y}} = 1 + \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot y^n = 1 - \frac{y}{2} + \frac{3}{8}\cdot y^2 - \cdots \quad \text{for } |y|<1.$$

$y=-x^2$ so $|y|<1$ becomes $|x^2|<1$ namely $|x|^2<1$ so $|x|<1$.

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot(-x^2)^n =$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot(-1)^n\cdot x^{2n} =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \cdot x^{2n}.$$