MATH 31B – SUMMER 2022

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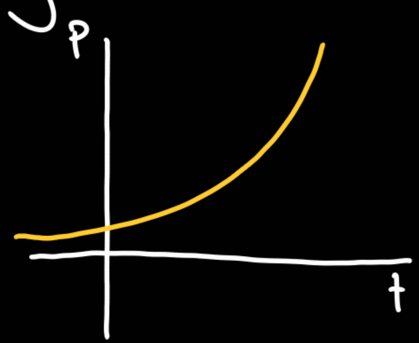
based on "Single Variable Calculus"
by Jonathan D. Rogawski.

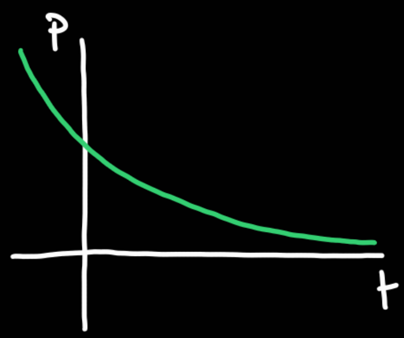
Section 7.4.: Exponential growth and decay.

Exponential growth: When a quantity $P(t)$ depends exponentially on time: $\boxed{P(t) = P_0 \cdot e^{kt}}$

growth constant: $k>0$



decay constant: $k<0$



To find P_0, set $t=0$: $P(0) = P_0 \cdot e^{k \cdot 0} = P_0 \cdot e^0 = P_0$.

Example: Population of bacteria. $k = 0.41$ hours^{-1}. 1000 bacteria at $t=0$.

a) Find $P(t)$.

$P_0 = P(0) = 1000$ so $P(t) = 1000 \cdot e^{0.41 \cdot t}$, t in hours.

b) How large is the population after 5 hours?

$P(5) = 1000 \cdot e^{0.41 \cdot 5} \approx 7767.9 \approx 7768$.

c) When will the population reach 10000?

b) When will the population reach 10000?

$10000 = P(t) = 1000 \cdot e^{0.41 \cdot t}$, $\quad e^{0.41 \cdot t} = 10$, $\quad 0.41 \cdot t = \ln(10)$,

$t = \dfrac{\ln(10)}{0.41} \approx 5.62$ hours, t is 5 hours and 37 minutes.

The exponential functions are the only functions satisfying the equation: $y' = k \cdot y$. Then $y(t) = P_0 \cdot e^{k \cdot t}$ where $P_0 = y(0)$.

y' is the <u>derivative</u> of y, also known as the <u>rate of change</u>.

<u>Example</u>: Penicillin leaves a person's bloodstream at a rate proportional to the amount present.

a) Express this as an equation.

$A(t)$ the quantity of penicillin in the bloodstream at time t.

$A(t) = -k \cdot A(t)$ with $k > 0$ because $A(t)$ is decreasing.

b) Find the decay constant if 50 mg of penicillin remain in the bloodstream 7 hours after an initial injection of 450 mg.

$A(7) = 50$, $A(0) = 450$, so:

$A(t) = 450 \cdot e^{-k \cdot t}$ and $50 = A(7) = 450 \cdot e^{-k \cdot 7}$ gives $k \approx 0.31$.

c) At what time were 200 mg present?

$200 = A(t) = 450 \cdot e^{-0.31 \cdot t}$ so $t \approx 2.62$ hours.

Doubling time: Time T such that $P(t)$ doubles in size: $P(t+T) = 2 \cdot P(t)$.

$P(t) = P_0 \cdot e^{k \cdot t}$, $k > 0$, then: $\boxed{T = \frac{\ln(2)}{k}}$

Example: Spread of a virus. $k = 0.0815\ s^{-1}$.

a) What is the doubling time?

$$T = \frac{\ln(2)}{0.0815} \approx 8.5 \text{ seconds.}$$

b) If the virus began in four individuals, how many hosts were infected after 2 minutes? And after 3 minutes?

$P_0 = P(0) = 4$, $P(t) = 4 \cdot e^{0.0815 \cdot t}$, 2 min = 120 seconds

$P(120) = 4 \cdot e^{0.0815 \cdot 120} \approx 70\,700.$ 3 min = 180 seconds

$P(180) = 4 \cdot e^{0.0815 \cdot 180} \approx 9\,400\,000.$

Half-life: Time T such that $P(t)$ halves in size: $P(t+T) = \frac{1}{2} \cdot P(t)$.

$P(t) = P_0 \cdot e^{-k \cdot t}$, $k > 0$, then: $\boxed{T = \frac{\ln(2)}{k}}$

Example: An isotope decays with a half life of 3.825 days. How long will it take for 80% of the isotope to decay?

$R(t) = R_0 \cdot e^{-k \cdot t}$, $3.825 = \dfrac{\ln(2)}{k}$ so $k = \dfrac{\ln(2)}{3.825} \approx 0.181$.

$R_0 = R(0)$ is the initial amount. When 80% has decayed, 20% remains,

so $R(t) = 0.2 \cdot R_0$: $R_0 \cdot e^{-0.181 \cdot t} = 0.2 \cdot R_0$, $t = \dfrac{\ln(0.2)}{-0.181} \approx 8.9$ days.

<u>Remark</u>: The formulas for the doubling time and the half-life are the same. For the doubling time we solve:

$$P(t+T) = 2 \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{k \cdot t}, \; k > 0.$$

$$P_0 \cdot e^{k \cdot (t+T)} = 2 \cdot P_0 \cdot e^{k \cdot t} \quad \text{so} \quad e^{k \cdot (t+T)} = 2 \cdot e^{k \cdot t}.$$

For the half-life we solve:

$$P(t+T) = \frac{1}{2} \cdot P(t) \quad \text{with } P(t) = P_0 \cdot e^{-k \cdot t}, \; k > 0$$

$$P_0 \cdot e^{-k \cdot (t+T)} = \frac{1}{2} \cdot P_0 \cdot e^{-k \cdot t} \quad \text{so} \quad \frac{1}{e^{k \cdot (t+T)}} = \frac{1}{2} \cdot \frac{1}{e^{k \cdot t}}$$

and the remaining equation is: $2 \cdot e^{k \cdot t} = e^{k \cdot (t+T)}$, the same equation as for the doubling time.

<u>Section 7.1</u>: Derivative of $f(x) = b^x$ and the number e.

<u>Exponential function</u>: $\boxed{f(x) = b^x}$ with <u>base</u> $b > 0$ and $b \neq 1$.

1. They are always strictly positive.

2. Their range is all the positive real numbers.

3. Increasing if $b>1$ and decreasing if $0<b<1$.

Laws of exponents:

Exponent zero	$b^0=1$
Products	$b^x b^y = b^{x+y}$
Quotients	$\frac{b^x}{b^y} = b^{x-y}$
Negative exponents	$b^{-x} = \frac{1}{b^x}$
Power to a power	$(b^x)^y = b^{xy}$
Roots	$b^{\frac{1}{n}} = \sqrt[n]{b}$

Example: Simplify:

a) $16^{-\frac{1}{2}} = \frac{1}{16^{\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$.

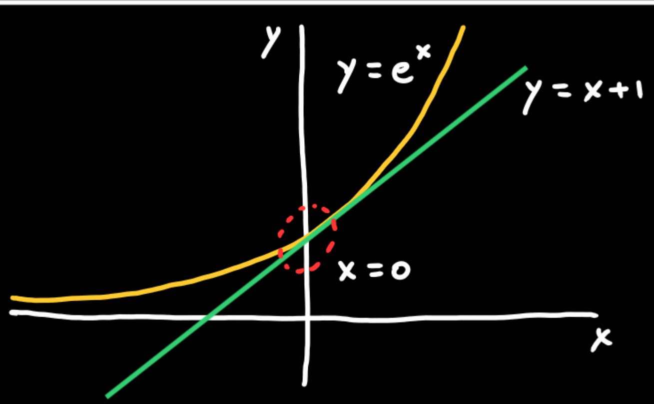
b) $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$.

c) $4^{16} \cdot 4^{-18} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$.

d) $\frac{9^3}{3^7} = \frac{(3^2)^3}{3^7} = \frac{3^6}{3^7} = 3^{-1} = \frac{1}{3}$.

Derivative of the exponential function: $\boxed{\frac{d}{dx}(b^x) = \ln(b) \cdot b^x}$

There is a unique positive real number e such that $\frac{d}{dx}(e^x) = e^x$



At $x=0$, the tangent line to e^x has slope $m=1$.

<u>Example:</u> Find the equation of the tangent line to $3e^x - 5x^2$ at $x=2$.

For $f(x) = 3e^x - 5x^2$ we have:

$$f'(x) = 3\cdot \frac{d}{dx}\left(e^x\right) - 5\cdot \frac{d}{dx}\left(x^2\right) = 3\cdot e^x - 10\cdot x,$$

$$f(2) = 3e^2 - 5\cdot(2^2) \approx 2.17$$

$$f'(2) = 3e^2 - 10\cdot 2 \approx 2.17$$

So the tangent line is $y = f(2) + f'(2)\cdot(x-2) \approx 2.17\cdot(x-1)$.

Using the chain rule for derivatives (with k and b constant):

$$\boxed{\frac{d}{dx}\left(e^{g(x)}\right) = g'(x)\cdot e^{g(x)} \quad \text{and} \quad \frac{d}{dx}\left(e^{kx+b}\right) = k\cdot e^{kx+b}}$$

<u>Example:</u> Differentiate:

a) $\frac{d}{dx}\left(e^{9x-5}\right) = 9\cdot e^{9x-5}$.

b) $\frac{d}{dx}\left(e^{\cos(x)}\right) = -(\sin(x))\cdot e^{\cos(x)}$.

<u>Integral of the exponential function:</u> (with k and b constant, $k \neq 0$)

$$\boxed{\int e^x \cdot dx = e^x + c_1 \quad \text{and} \quad \int e^{kx+b} \cdot dx = \frac{1}{k} \cdot e^{kx+b} + c_1}$$

Example: Evaluate:

a) $\int e^{7x-5} \cdot dx = \frac{1}{7} \cdot e^{7x-5} + c_1.$

b) $\int x \cdot e^{2x^2} \cdot dx = \frac{1}{4} \int e^u \, du = \frac{1}{4} e^u + c_1 = \frac{1}{4} e^{2x^2} + c_1.$

$u = 2x^2$
$du = 4x\,dx$

c) $\int \frac{e^t}{1+2e^t+e^{2t}} \cdot dt = \int \frac{e^t}{(1+e^t)^2} \cdot dt = \int \frac{du}{(1+u)^2} = -(1+u)^{-1} + c_1 =$

$u = e^t$
$du = e^t dt$

$= -(1+e^t)^{-1} + c_1.$

Section 7.2: Inverse functions.

The inverse of $f(x)$, is the function that reverses $f(x)$.

Let $f(x)$ have domain D and range R. If there is a function $g(x)$ with domain R such that $g(f(x)) = x$ for all $x \in D$ and $f(g(x)) = x$ for all $x \in R$ then $f(x)$ is said to be invertible. We call $g(x)$ the inverse, and is denoted $f^{-1}(x)$.

Example: Find the inverse of $f(x) = 2x - 18$.

Step 1: Solve $y = f(x)$ for x in terms of y.

$$y = 2x - 18 \quad \text{so} \quad y + 18 = 2x \quad \text{so} \quad x = \frac{y}{2} + 9.$$

Thus $f^{-1}(y) = \frac{y}{2} + 9$.

Step 2: Rewrite with x instead of y. $f^{-1}(x) = \frac{x}{2} + 9$.

Step 3: Check $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$f^{-1}(f(x)) = f^{-1}(2x - 18) = \frac{2x-18}{2} + 9 = x - 9 + 9 = x.$$

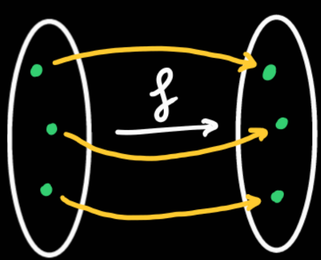
$$f(f^{-1}(x)) = f\left(\frac{x}{2} + 9\right) = 2 \cdot \left(\frac{x}{2} + 9\right) - 18 = x + 18 - 18 = x.$$

If $f^{-1}(x)$ exists, it is unique. However, some functions like $f(x) = x^2$ do not have an inverse. When is a given function invertible?

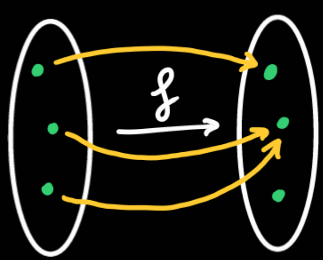
> A function $f(x)$ is one-to-one on a domain D if for every $c \in D$ the equation $f(x) = c$ has at most one solution $x \in D$.

Equivalently, if $f(a) = f(b)$ then $a = b$.

one-to-one:



not one-to-one:



The inverse function $f^{-1}(x)$ exists if and only if $f(x)$ is one-to-one on its domain D. Then the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.

<u>Example:</u> Find the inverse of $f(x) = \dfrac{3x+2}{5x-1}$.

The domain of $f(x)$ is $D = \{x \mid x \neq \frac{1}{5}\}$. For $x \in D$, solve $y = f(x)$ for x.

$y = \dfrac{3x+2}{5x-1}$ so $(5x-1)y = 3x-2$ so $5xy - y = 3x-2$ so

$5xy - 3x = y+2$ so $x(5y-3) = y+2$ so $x = \dfrac{y+2}{5y-3}$

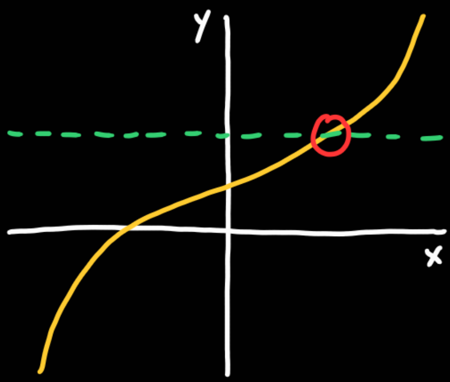
whenever $y \neq \frac{3}{5}$. However $y = \frac{3}{5}$ is <u>not</u> in the range of $f(x)$ since otherwise $x(5y-3) = y+2$ gives $0 = \frac{3}{5} + 2$, which is false.

Since $x = \dfrac{y+2}{5y+3}$, for each $y \neq \frac{3}{5}$ there is a unique x with $f(x) = y$.

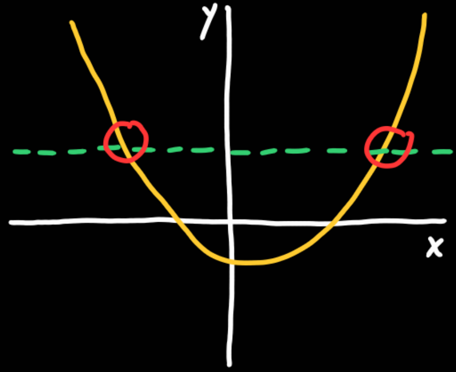
So $f(x)$ is one-to-one on its domain, so it is invertible. The range of $f(x)$ is $R = \{x \mid x \neq \frac{3}{5}\}$ and $f^{-1}(x) = \dfrac{x+2}{5x-3}$, which has range D and domain R.

<u>Horizontal line test:</u> A function $f(x)$ is one-to-one if and only if every horizontal line intersects the graph of $f(x)$ in at most one point.

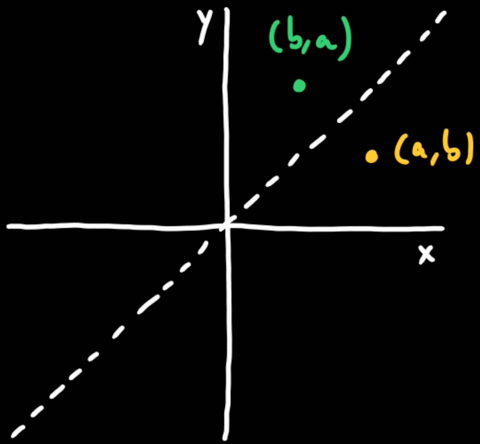
one-to-one:

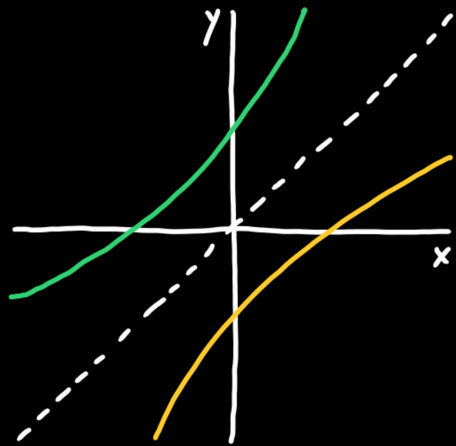


not one-to-one:



$$\boxed{\text{The graph of } f^{-1} \text{ is the reflection of the graph of } f \text{ through } y = x.}$$





Derivative of the inverse:

$$\boxed{\left(f^{-1}(b)\right)' = \frac{1}{f'\left(f^{-1}(b)\right)}}$$

$f(x)$ differentiable and one-to-one, b in domain of $f^{-1}(x)$, $f'(f^{-1}(b)) \neq 0$.

Example: Calculate $\left(f^{-1}(x)\right)'$ for $f(x) = x^4 + 10$ on $D = \{x \mid x \geq 0\}$.

Solve $y = x^4 + 10$ for x to obtain $x = (y-10)^{\frac{1}{4}}$, so $f^{-1}(x) = (x-10)^{\frac{1}{4}}$.

Now $f'(x) = 4x^3$ so $f'\left(f^{-1}(x)\right) = 4 \cdot \left(f^{-1}(x)\right)^3 = 4 \cdot (x-10)^{\frac{3}{4}}$ so:

$$\left(f^{-1}(x)\right)' = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{4 \cdot (x-10)^{\frac{3}{4}}} = \frac{(x-10)^{-\frac{3}{4}}}{4}.$$

If we directly differentiate $f'(x)$ we also obtain this.

Section 7.3: Logarithms and their derivatives.

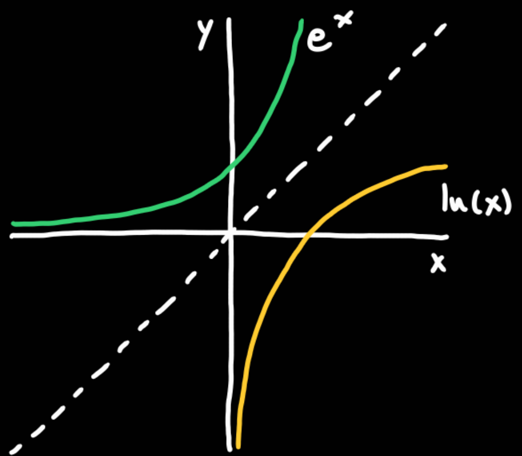
Logarithms are inverses of exponentials.

$$\boxed{b^{\log_b(x)} = x \qquad \text{and} \qquad \log_b(b^x) = x}$$

Thus $\log_b(x)$ is the number to which b must be raised to get x.

1. The domain of $\log_b(x)$ is $\{x \mid x>0\}$.

2. The range of $\log_b(x)$ is all real numbers.



If $b>1$ then $\log_b(x)>0$ for $x>1$, $\log_b(x)<0$ for $x<1$, and:

$$\boxed{\lim_{x\to 0^+} \log_b(x) = -\infty \qquad , \qquad \lim_{x\to\infty} \log_b(x) = \infty}$$

Laws of logarithms:

Log of 1	$\log_b(1) = 0$
Log of b	$\log_b(b) = 1$

Products	$\log_b(x\cdot y) = \log_b(x) + \log_b(y)$
Quotients	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
Reciprocals	$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$
Powers	$\log_b(x^u) = u\cdot\log_b(x)$

Change of base:

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}, \quad \log_b(x) = \frac{\ln(x)}{\ln(b)}}$$

Example: Evaluate:

a) $\log_6(9) + \log_6(4) = \log_6(9\cdot 4) = \log_6(36) = \log_6(6^2) = 2.$

b) $\ln\left(\frac{1}{\sqrt{e}}\right) = \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2}\ln(e) = -\frac{1}{2}.$

c) $10\cdot\log_b(b^3) - 4\cdot\log_b(\sqrt{b}) = 10\cdot 3 - 4\cdot\log_b\left(b^{\frac{1}{2}}\right) = 30 - 4\cdot\frac{1}{2} = 28.$

Derivative of the exponential function:

$$\boxed{\frac{d}{dx}\left(b^x\right) = \ln(b)\cdot b^x}$$

Example: Differentiate:

a) $\frac{d}{dx}\left(4^{3x}\right) = \frac{d}{du}\left(4^u\right)\cdot\frac{d}{dx}(u) = \ln(4)\cdot 4^u\cdot 3 = 3\cdot\ln(4)\cdot 4^{3x}$

$u = 3x$

$du = 3\,dx$

b) $\frac{d}{dx}\left(5^{x^2}\right) = \frac{d}{du}\left(5^u\right)\cdot\frac{d}{dx}(u) = \ln(5)\cdot 5^u\cdot 2\cdot x = 2\cdot\ln(5)\cdot x\cdot 5^{x^2}.$

$u = x^2$
$du = 2x\,dx$

Derivative of the natural logarithm:

$$\boxed{\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x} \quad , x>0}$$

Example: Differentiate:

a) $\frac{d}{dx}\left(x\cdot\ln(x)\right) = x\cdot\frac{d}{dx}(\ln(x)) + \frac{d}{dx}(x)\cdot\ln(x) = x\cdot\frac{1}{x} + \ln(x) = 1+\ln(x).$

b) $\frac{d}{dx}\left(\ln(x)^2\right) = 2\cdot\ln(x)\cdot\frac{d}{dx}\left(\ln(x)\right) = \frac{2\cdot\ln(x)}{x}.$

Derivative of log composite:

$$\boxed{\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}}$$

Example: Differentiate:

a) $\frac{d}{dx}\left(\ln(x^3+1)\right) = \frac{3x^2}{x^3+1}.$

b) $\frac{d}{dx}\left(\ln\left(\sqrt{\sin(x)}\right)\right) = \frac{d}{dx}\left(\ln\left(\sin(x)^{\frac{1}{2}}\right)\right) = \frac{1}{2}\cdot\frac{d}{dx}\left(\ln(\sin(x))\right) =$

$= \frac{\cos(x)}{2\cdot\sin(x)}$

c) $\frac{d}{dx}\left(\log_{10}(x)\right) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(10)}\right) = \frac{1}{\ln(10)}\cdot\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{\ln(10)\cdot x}.$

d) $\frac{d}{dx}\left(\frac{(x+1)^2\cdot(2x^2-3)}{\sqrt{x^2+1}}\right) = \frac{\frac{d}{dx}\left(f(x)\cdot g(x)\right)\cdot h(x) - f(x)\cdot g(x)\cdot\frac{d}{dx}\left(h(x)\right)}{h(x)^2} =$

$f(x) = (x+1)^2, \quad g(x) = 2x^2-3, \quad h(x) = \sqrt{x^2+1}.$

$f'(x) = 2(x+1), \quad g'(x) = 4x, \quad h'(x) = \frac{x}{\sqrt{x^2+1}}$

$\left(f'(x)\cdot g(x) + f(x)\cdot g'(x)\right)\cdot h(x) - f(x)\cdot g(x)\cdot h'(x)$

$$= \frac{\quad}{h(x)^2} = \dots =$$

$$= \frac{6x^5+8x^4+7x^3+12x^2+x-6}{(x^2+1)^{\frac{3}{2}}}.$$

<u>Logarithmic differentiation:</u> Differentiate $\ln(f(x))$:

$$\ln(f(x)) = \ln((x+1)^2) + \ln(2x^2-3) - \ln\left(\sqrt{x^2+1}\right) =$$

$$= 2\cdot\ln(x+1) + \ln(2x^2-3) - \frac{1}{2}\cdot\ln(x^2+1)$$

Then:

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\left(\ln(f(x))\right) = 2\cdot\frac{d}{dx}\left(\ln(x+1)\right) + \frac{d}{dx}\left(\ln(2x^2-3)\right) - \frac{1}{2}\cdot\frac{d}{dx}\left(\ln(x^2+1)\right) =$$

$$= \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{1}{2}\cdot\frac{2x}{x^2+1}$$

So multiplying by $f(x)$:

$$f'(x) = \left(\frac{(x+1)^2\cdot(2x^2-3)}{\sqrt{x^2+1}}\right)\cdot\left(\frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1}\right) =$$

$$= \frac{6x^5+8x^4+7x^3+12x^2+x-6}{(x^2+1)^{\frac{3}{2}}}.$$

Section 7.7.: L'Hôpital's rule.

L'Hôpital's rule is a tool for computing limits and determining "asymptotic behavior"; that is, limits at infinity.

<u>L'Hôpital's rule:</u> Assume that $f(x)$ and $g(x)$ are differentiable around a and that $f(a)=0=g(a)$. Assume also that $g'(x)\neq 0$ except possibly at $x=a$.

Then if the limit exists:

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.}$$

This also holds if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, and it is valid for one-sided limits.

<u>Example</u>: Evaluate:

$$\lim_{x\to 2} \frac{x^3-8}{x^4+2x-20} = \lim_{x\to 2} \frac{3x^2}{4x^3+2} = \frac{3\cdot 4}{4\cdot 8+2} = \frac{12}{34} = \frac{6}{17}.$$

$f(x) = x^3-8 \qquad f(2)=0$

$g(x) = x^4+2x-20 \qquad g(2)=0 \qquad g'(x) = 4x^3+2$ is <u>not</u> zero near $x=2$

<u>Example</u>: Evaluate:

$$\lim_{x\to 2} \frac{4-x^2}{\sin(\pi x)} = \lim_{x\to 2} \frac{-2x}{\pi\cdot\cos(\pi x)} = \frac{-2\cdot 2}{\pi\cdot\cos(2\pi)} = \frac{-4}{\pi}.$$

$f(x) = 4-x^2 \qquad f(2)=0$

$g(x) = \sin(\pi x) \qquad g(2)=0 \qquad g'(x) = \pi\cdot\cos(\pi x)$ is <u>not</u> zero near $x=2$.

<u>Example</u>: Evaluate:

$$\lim_{x\to \frac{\pi}{2}} \frac{\cos^2(x)}{1-\sin(x)} = \lim_{x\to \frac{\pi}{2}} \frac{-2\cdot\sin(x)\cdot\cos(x)}{-\cos(x)} = \lim_{x\to\frac{\pi}{2}} 2\cdot\sin(x) = 2.$$

$f(x) = \cos^2(x) \qquad f(\frac{\pi}{2})=0$

$g(x) = 1-\sin(x) \qquad g(\frac{\pi}{2})=0 \qquad g'(x) = -\cos(x)$ is <u>not</u> zero near $x=\frac{\pi}{2}$.

Example: Evaluate:

$$\lim_{x\to 0^+} x\cdot\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

$f(x) = x$ $\quad$ $f(x) \to 0$ $\quad$ $f(x) = \frac{1}{x}$

$g(x) = \ln(x)$ $\quad$ $g(x) \to -\infty$ $\quad$ $g(x) = \ln(x)$ $\quad$ L'Hôpital's Rule applies.

Example: Evaluate:

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos(x) - 1} = \lim_{x\to 0} \frac{e^x - 1}{-\sin(x)} = \lim_{x\to 0} \frac{e^x}{-\cos(x)} = -1.$$

$f(x) = e^x - x - 1$ $\quad$ $f(x) = e^x - 1$

$g(x) = \cos(x) - 1$ $\quad$ $g(x) = -\sin(x)$

Example: Evaluate:

$$\lim_{x\to 0} \left(\frac{1}{\sin(x)} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin(x)}{x\cdot\sin(x)} = \lim_{x\to 0} \frac{1 - \cos(x)}{x\cdot\cos(x) + \sin(x)} =$$

$f(x) = x - \sin(x)$ $\quad$ $f(x) = 1 - \cos(x)$

$g(x) = x\cdot\sin(x)$ $\quad$ $g(x) = x\cdot\cos(x) - \sin(x)$

$$= \lim_{x\to 0} \frac{\sin(x)}{-x\cdot\sin(x) + 2\cdot\cos(x)} = 0.$$

Example: Evaluate:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = e^{\lim_{x\to 0^+} \ln(x^x)} = e^0 = 1$$

$f(x) = e^x$ is continuous $\quad$ ⊛

⊛ $\lim \ln(x^x) = \lim x\cdot\ln(x) = 0$ as we have seen above.

$x\to 0^+$ $\qquad$ $x\to 0^+$

> We say that $f(x)$ grows faster than $g(x)$ if:
>
> $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty \quad \text{or} \quad \lim_{x\to\infty}\frac{g(x)}{f(x)}=0 \quad \text{and denote } f(x) >> g(x).$$

<u>L'Hôpital's rule:</u> Assume that $f(x)$ and $g(x)$ are differentiable in an interval (b,∞). Assume also that $g'(x)\neq 0$ for $x>b$. If $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist and are either both infinite or zero, then:

$$\boxed{\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}}$$

whenever the limit exists. This also holds for $x\to-\infty$.

<u>Example:</u> Which of $f(x)=x^2$ and $g(x)=x\cdot\ln(x)$ grows faster?

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^2}{x\cdot\ln(x)}\underset{\text{LHR}}{=}\lim_{x\to\infty}\frac{x}{\ln(x)}\underset{\text{LHR}}{=}\lim_{x\to\infty}\frac{1}{\frac{1}{x}}=\lim_{x\to\infty} x=\infty$$

so $f(x)$ grows faster.

<u>Example:</u> Evaluate:

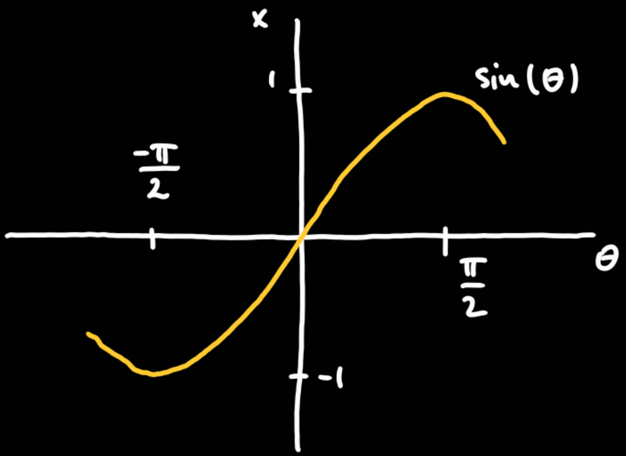
$$\lim_{x\to\infty}\frac{\sqrt{x}}{\ln(x)^2}\underset{\text{LHR}}{=}\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{2}{x}\cdot\ln(x)}=\lim_{x\to\infty}\frac{\sqrt{x}}{4\cdot\ln(x)}\underset{\text{LHR}}{=}\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}}=$$

$$=\lim_{x\to\infty}\frac{\sqrt{x}}{8}=\infty.$$

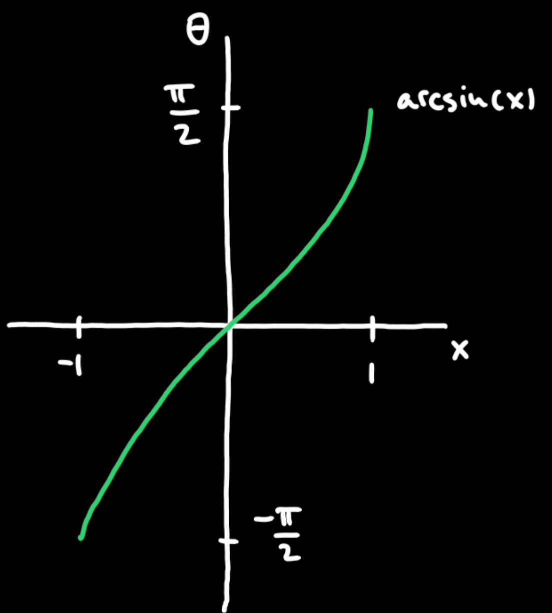
Growth rule of thumb: $\boxed{e^x \gg x^n \gg \ln(x), \quad n \text{ integer.}}$

Section 7.8.: Inverse trigonometric functions.

The function $f(\theta) = \sin(\theta)$ is one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



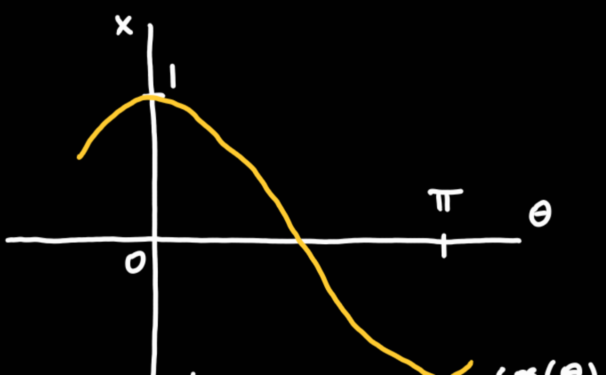
Its inverse is called the arcsine function, denoted $\boxed{\arcsin(x).}$

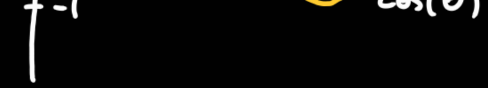


Domain: $[-1, 1]$.

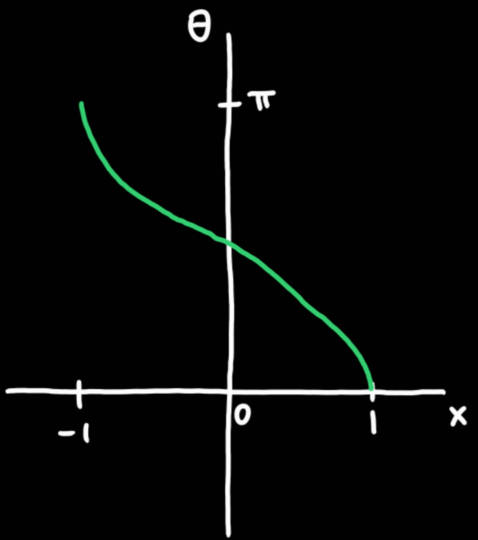
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The function $f(\theta) = \cos(\theta)$ is one-to-one on $[0, \pi]$.





Its inverse is called the arccosine function, denoted $\boxed{\arccos(x).}$



Domain: $[-1,1]$.

Range: $[0,\pi]$.

<u>Derivatives of arcsine and arccosine:</u>

$$\boxed{\frac{d}{dx}\left(\arcsin(x)\right)=\frac{1}{\sqrt{1-x^2}}\ ,\quad \frac{d}{dx}\left(\arccos(x)\right)=\frac{-1}{\sqrt{1-x^2}}}$$

<u>Example:</u> $\frac{d}{dx}\left(\arcsin(x^2)\right)=\frac{1}{\sqrt{1-x^4}}\cdot\frac{d}{dx}(x^2)=\frac{2x}{\sqrt{1-x^4}}$.

The function $f(\theta)=\tan(\theta)$ is one-to-one on $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$. Its inverse is called the arctangent function, denoted $\boxed{\arctan(x).}$

The function $f(\theta)=\cot(\theta)$ is one-to-one on $(0,\pi)$. Its inverse is called the arccotangent function, denoted $\boxed{\text{arccotan}(x).}$

The function $f(\theta)=\sec(\theta)$ is one-to-one on $\left[0,\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2},\pi\right]$. Its inverse is called the arcsecant function, denoted $\boxed{\operatorname{arcsec}(x).}$

The function $f(\theta)=\csc(\theta)$ is one-to-one on $\left[\frac{-\pi}{2},0\right)$ and $\left(0,\frac{\pi}{2}\right]$. Its inverse

is called the arccosecant function, denoted $\boxed{\operatorname{arccsc}(x).}$

<u>Derivatives of inverse trigonometric functions:</u>

$$\boxed{\begin{aligned}&\frac{d}{dx}\left(\arctan(x)\right)=\frac{1}{x^2+1}, &&\frac{d}{dx}\left(\operatorname{arccot}(x)\right)=\frac{-1}{x^2+1},\\ &\frac{d}{dx}\left(\operatorname{arcsec}(x)\right)=\frac{1}{|x|\sqrt{x^2-1}}, &&\frac{d}{dx}\left(\operatorname{arccsc}(x)\right)=\frac{-1}{|x|\sqrt{x^2-1}}.\end{aligned}}$$

<u>Example:</u> Integrate:

$$\int_0^1 \frac{dx}{x^2+1} = \arctan(x)\Big|_0^1 = \arctan(1)-\arctan(0) = \frac{\pi}{4}-0=\frac{\pi}{4}.$$

<u>Example:</u> Integrate:

$$\int_{\frac{1}{\sqrt{2}}}^1 \frac{dx}{x\sqrt{4x^2-1}} = \int_{\sqrt{2}}^2 \frac{\frac{1}{2}du}{\frac{1}{2}u\sqrt{u^2-1}} = \int_{\sqrt{2}}^2 \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}(u)\Big|_{\sqrt{2}}^2 =$$

$u=2x$ $\quad$ $x=1 \rightsquigarrow u=2$

$du=2dx$ $\quad$ $x=\frac{1}{\sqrt{2}} \rightsquigarrow u=\sqrt{2}$

$$=\operatorname{arcsec}(2)-\operatorname{arcsec}(\sqrt{2})=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}.$$

<u>Example:</u> Integrate:

$$\int_{\frac{-3}{4}}^0 \frac{dx}{\sqrt{9-16x^2}} = \int_{\frac{-3}{4}}^0 \frac{dx}{3\cdot\sqrt{1-\left(\frac{4x}{3}\right)^2}} = \int_{-1}^0 \frac{\frac{3}{4}du}{3\sqrt{1-u^2}} = \frac{1}{4}\int_{-1}^0 \frac{du}{\sqrt{1-u^2}} =$$

$$\sqrt{9-16x^2}=\sqrt{9\cdot\left(1-\frac{16x^2}{9}\right)}=3\sqrt{1-\left(\frac{4x}{3}\right)^2}$$

$u=\frac{4x}{3}$ $\quad$ $u(0)=0$

$du=\frac{4dx}{3}$ $\quad$ $u\left(\frac{-3}{4}\right)=-1$

$$=\frac{1}{4}\arcsin(u)\Big|_{-1}^0=\frac{1}{4}\left(\arcsin(0)-\arcsin(-1)\right)=\frac{1}{4}\left(0-\left(\frac{-\pi}{2}\right)\right)=\frac{\pi}{8}$$

$$= \frac{\;}{4}\arcsin(x)\Big|_{-1} = \frac{\;}{4}\left(\arcsin(0)-\arcsin(-1)\right) = \frac{\;}{4}\left(0-\left(\frac{\;}{2}\right)\right) = \frac{\;}{8}.$$

Section 7.9: Hyperbolic functions.

The hyperbolic functions are specific combinations of e^x and e^{-x}.

Hyperbolic sine: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

Hyperbolic cosine: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

Hyperbolic tangent: $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent: $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant: $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$

Hyperbolic cosecant: $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)} = \dfrac{2}{e^x - e^{-x}}$

Derivatives of hyperbolic functions:

$$\frac{d}{dx}\left(\sinh(x)\right) = \cosh(x), \qquad \frac{d}{dx}\left(\cosh(x)\right) = \sinh(x)$$

$$\frac{d}{dx}\left(\tanh(x)\right) = \operatorname{sech}^2(x), \qquad \frac{d}{dx}\left(\coth(x)\right) = -\operatorname{csch}^2(x)$$

$$\frac{d}{dx}\left(\operatorname{sech}(x)\right) = -\operatorname{sech}(x)\cdot\tanh(x), \qquad \frac{d}{dx}\left(\operatorname{csch}(x)\right) = -\operatorname{csch}(x)\cdot\coth(x).$$

Example: Simplify:

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} + 2}{\;}$$

$$-\frac{e^{2x}+e^{-2x}-2}{4}=\frac{2}{4}+\frac{2}{4}=1.$$

Example: Differentiate:

$$\frac{d}{dx}\left(\coth(x)\right)=\frac{d}{dx}\left(\frac{\cosh(x)}{\sinh(x)}\right)=\frac{\frac{d}{dx}\left(\cosh(x)\right)\cdot\sinh(x)-\cosh(x)\cdot\frac{d}{dx}\left(\sinh(x)\right)}{\sinh^2(x)}=$$

$$=\frac{\sinh^2(x)-\cosh^2(x)}{\sinh^2(x)}=\frac{-1}{\sinh^2(x)}=-\operatorname{csch}^2(x).$$

Inverse hyperbolic functions and their derivatives:

Function	Domain	Derivative
$\operatorname{arcsinh}(x)$	$\mathbb{R}$	$\frac{1}{\sqrt{x^2+1}}$
$\operatorname{arccosh}(x)$	$[1,\infty)$	$\frac{1}{\sqrt{x^2-1}}$
$\operatorname{arctanh}(x)$	$(-1,1)$	$\frac{1}{1-x^2}$
$\operatorname{arccoth}(x)$	$(-\infty,-1)\cup(1,\infty)$	$\frac{1}{1-x^2}$
$\operatorname{arcsech}(x)$	$(0,1]$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{arccsch}(x)$	$(-\infty,0)\cup(0,\infty)$	$\frac{-1}{\lvert x\rvert\sqrt{x^2+1}}$

Example: Differentiate:

$$\frac{d}{dx}\left(\operatorname{arctanh}(x)\right)=\frac{1}{\operatorname{sech}^2(\operatorname{arctanh}(x))}=\frac{1}{1-x^2}.$$

if $g(x)$ is the inverse of $f(x)$,

$1=\cosh^2(t)-\sinh^2(t)$

then $g'(x) = \dfrac{1}{f'(g(x))}$.

$f(x) = \tanh(x)$, $f'(x) = \operatorname{sech}^2(x)$

$g(x) = \operatorname{arctanh}(x)$

$$\frac{1}{\cosh^2(t)} = 1 - \frac{\sinh^2(t)}{\cosh^2(t)}$$

$$\operatorname{sech}^2(t) = 1 - \tanh^2(t)$$

$t = \operatorname{arctanh}(x)$

$$\operatorname{sech}^2(\operatorname{arctanh}(x)) = 1 - x^2$$

Section 8.1: Integration by parts.

The formula for integration by parts is given by the product rule for differentiation: $\frac{d}{dx}\left(u(x)\cdot v(x)\right) = \frac{d}{dx}\left(u(x)\right)\cdot v(x) + u(x)\cdot \frac{d}{dx}\left(v(x)\right)$, so:

Integration by parts: $$\boxed{\int u(x)\, v'(x)\,dx = u(x)\,v(x) - \int v(x)\cdot u'(x)\,dx}$$

Or in shorthand: $\int u\cdot dv = u\cdot v - \int v\cdot du$. Guidelines for choosing u and v:

1. We want $\frac{du}{dx}$ simpler than u.

2. We need to know how to evaluate $\int v'(x)\,dx$ to compute v.

Example: Evaluate:

$$\int x\cdot\cos(x)\,dx = x\cdot\sin(x) - \int \sin(x)\,dx = x\cdot\sin(x) + \cos(x) + C$$

$u = x$ $\qquad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos(x)$ $\qquad v = \sin(x)$

Example: Evaluate:

$$\int x\,e^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

$u = x$ $\qquad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^x$ $\qquad v = e^x$

$$\frac{\ }{dx} = e \qquad v = e$$

However, if we swap our choices:

$$\int x e^x\,dx = \frac{x^2}{2}\cdot e^x - \int \frac{x^2}{2}\cdot e^x\,dx$$, which is a harder integral than the original.

⚠

$$u = e^x \qquad \frac{du}{dx} = e^x$$
$$\frac{dv}{dx} = x \qquad v = \frac{x^2}{2}$$

__Example:__ Evaluate:

$$\int x^2\cdot\cos(x)\,dx = x^2\cdot\sin(x) - \int 2x\sin(x)\,dx = x^2\sin(x) - 2\int x\sin(x)\,dx =$$

$$u = x^2 \qquad \frac{du}{dx} = 2x$$
$$\frac{dv}{dx} = \cos(x) \qquad v = \sin(x)$$

$$u = x \qquad \frac{du}{dx} = 1$$
$$\frac{dv}{dx} = \sin(x) \qquad v = -\cos(x)$$

$$= x^2\cdot\sin(x) - 2\cdot\left(-x\cos(x) - \int 1\cdot(-\cos(x))\,dx\right) =$$

$$= x^2\cdot\sin(x) + 2x\cos(x) - 2\int\cos(x)\,dx = x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C.$$

__Example:__ Evaluate:

$$\int e^x\cdot\cos(x)\,dx = e^x\cos(x) - \int e^x\cdot(-\sin(x))\,dx = e^x\cos(x) + \int e^x\sin(x)\,dx =$$

$$u = \cos(x) \qquad \frac{du}{dx} = -\sin(x)$$
$$\frac{dv}{dx} = e^x \qquad v = e^x$$

$$u = \sin(x) \qquad \frac{du}{dx} = \cos(x)$$
$$\frac{dv}{dx} = e^x \qquad v = e^x$$

$$= e^x\cos(x) + \left(e^x\sin(x) - \int e^x\cos(x)\,dx\right) = e^x\cdot(\cos(x) + \sin(x)) - \int e^x\cos(x)\,dx$$

So:

$$2\cdot\int e^x\cdot\cos(x)\,dx = e^x\cdot(\cos(x) + \sin(x))$$

$$\int e^x \cos(x)\,dx = \frac{e^x}{2}\left(\cos(x)+\sin(x)\right) + C.$$

Integration by parts for definite integrals:

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

Example: Evaluate:

$$\int_1^3 \ln(x)\,dx = x\ln(x)\Big|_1^3 - \int_1^3 x\cdot\frac{1}{x}\cdot dx = x\ln(x)\Big|_1^3 - \int_1^3 dx = x\ln(x)\Big|_1^3 - x\Big|_1^3 =$$

$u=\ln(x)$ $\quad \frac{du}{dx}=\frac{1}{x}$

$\frac{dv}{dx}=1$ $\quad v=x$

$$= (3\ln(3)-0)-(3-1) = 3\ln(3)-2.$$

Example: Evaluate:

$$\int x^n e^x\,dx = x^n e^x - \int n x^{n-1} e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx.$$

$u=x^n$ $\quad \frac{du}{dx}=nx^n$

$\frac{dv}{dx}=e^x$ $\quad v=e^x$

Example: Evaluate:

$$\int x^3e^x\,dx = x^3e^x - 3\int x^2e^x\,dx = x^3e^x - 3\left(x^2e^x - 2\int xe^x\,dx\right) = x^3e^x - 3x^2e^x + 6\int xe^x\,dx =$$

$$= x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x\,dx\right) = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C =$$

$$= (x^3-3x^2+6x-6)e^x + C.$$

Section 8.5: The method of partial fractions

When integrating a function $f(x)=\frac{P(x)}{Q(x)}$, we should try to rewrite $f(x)$ as a

sum of simpler fractions that can be integrated directly.

sum of simpler fractions that can be integrated directly.

Example: Evaluate:

$$\int \frac{dx}{x^2-1} = \int \frac{dx}{(x-1)(x+1)} = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{\ln|x-1|}{2} - \frac{\ln|x+1|}{2}$$

$$\frac{1}{(x-1)(x+1)} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

If the degree of $P(x)$ is less than the degree of $Q(x)$ and $Q(x)$ factors as a product of distinct linear factors: $Q(x) = (x-a_1)\cdots(x-a_n)$, then there is a partial fraction decomposition: $\dfrac{P(x)}{Q(x)} = \dfrac{A_1}{x-a_1} + \cdots + \dfrac{A_n}{x-a_n}$.

Each distinct factor $x-a$ in the denominator contributes a term $\dfrac{A}{x-a}$ to the partial fraction decomposition.

Example: Decompose into partial fractions:

$$\frac{5x^2+x-28}{x^3-4x^2+x+6} = \frac{5x^2+x-28}{(x+1)(x-2)(x-3)} = \frac{-2}{x+1} + \frac{2}{x-2} + \frac{5}{x-3}.$$

Example: Decompose into partial fractions and integrate $\dfrac{x^2+2}{2x^3-6x^2-12x+16}$.

We first factor the denominator: $2x^3-6x^2-12x+16 = (x-1)(2x-8)(x+2)$.

Then we write the decomposition: $\dfrac{x^2+2}{2x^3-6x^2-12x+16} = \dfrac{A}{x-1} + \dfrac{B}{2x-8} + \dfrac{C}{x+2}$.

Multiply by $(x-1)(2x-8)(x+2)$ to clear denominators:

$$x^2+2 = A\cdot(2x-8)(x+2) + B\cdot(x-1)(x+2) + C\cdot(x-1)(2x-8).$$

To compute A, set $x=1$: $1^2+2 = A\cdot(2-8)(1+2)$ so $A = \frac{-1}{6}$.

To compute B, set $x=4$: $4^2+2 = B\cdot(4-1)(4+2)$ so $B=1$.

To compute C, set $x=-2$: $(-2)^2+2 = C\cdot(-2-1)(-4-8)$ so $C=\frac{1}{6}$.

Then:
$$\frac{x^2+2}{2x^3-6x^2-12x+16} = \frac{\frac{-1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2}.$$

We can then integrate:

$$\int \frac{x^2+2}{2x^3-6x^2-12x+16}\,dx = \frac{-1}{6}\int\frac{dx}{x-1} + \int\frac{dx}{2x-8} + \frac{1}{6}\int\frac{dx}{x+2} =$$

$$= \frac{-1}{6}\ln|x-1| + \frac{1}{2}\ln|x-4| + \frac{1}{6}\ln|x+2| + C.$$

<u>Remark</u>: We can also factor: $2x^3-6x^2-12x+16 = 2(x-1)(x-4)(x+2)$ and use the decomposition:

$$\frac{x^2+2}{2x^3-6x^2-12x+16} = \frac{x^2+2}{2(x-1)(x-4)(x+2)} = \frac{D}{x-1} + \frac{E}{x-4} + \frac{F}{x+2}$$

to obtain the same final integral.

If the degree of $P(x)$ is less than the degree of $Q(x)$ and $Q(x)$ factors as a product of repeated linear factors: $Q(x) = (x-a_1)^{M_1}\cdots(x-a_n)^{M_n}$, then there is a partial fraction decomposition:

$$\frac{P(x)}{Q(x)} = \frac{A_{11}}{x-a_1} + \frac{A_{12}}{(x-a_1)^2} + \cdots + \frac{A_{1M_1}}{(x-a_1)^{M_1}} + \cdots + \frac{A_{11}}{x-a_n} + \frac{A_{12}}{(x-a_n)^2} + \cdots + \frac{A_{1M_n}}{(x-a_n)^{M_n}}$$

Example: Decompose into partial fractions and integrate $\frac{3x-9}{x^3+3x^2-4}$.

We first factor the denominator: $x^3+3x^2-4=(x-1)(x+2)^2$.

Then we write the decomposition: $\frac{3x-9}{x^3+3x^2-4}=\frac{A}{x-1}+\frac{B_1}{x+2}+\frac{B_2}{(x+2)^2}$.

Multiply by $(x-1)(x+2)^2$ to clear denominators:

$$3x-9=A(x+2)^2+B_1(x-1)(x+2)+B_2(x-1).$$

To compute A, set $x=1$: $3\cdot 1-9=A(1+2)^2$ so $A=\frac{-2}{3}$.

To compute B_2, set $x=-2$: $3\cdot(-2)-9=B_2(-2-1)$ so $B_2=5$.

To compute B_1, set $x=2$: $3\cdot 2-9=\frac{-2}{3}\cdot(2+2)^2+B_1(2-1)(2+2)+5(2-1)$ so $B_1=\frac{2}{3}$.

Then: $\frac{3x-9}{x^3+3x^2-4}=\frac{\frac{-2}{3}}{x-1}+\frac{\frac{2}{3}}{x+2}+\frac{5}{(x+2)^2}$.

We can then integrate:

$$\int\frac{3x-9}{x^3+3x^2-4}dx=\frac{-2}{3}\int\frac{dx}{x-1}+\frac{2}{3}\int\frac{dx}{x+2}+5\int\frac{dx}{(x+2)^2}=$$

$$=\frac{-2}{3}\ln|x-1|+\frac{2}{3}\ln|x+2|+\frac{-5}{x+2}+C.$$

A power $(ax^2+bx+c)^M$ of a quadratic polynomial ax^2+bx+c that cannot be written as a product of two linear factors contributes to a partial fraction decomposition with:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\dots+\frac{A_Mx+B_M}{(ax^2+bx+c)^M}$$

$ax^2+bx+c \qquad (ax^2+bx+c) \qquad (ax^2+bx+c)$

Example: Decompose into partial fractions:

$$\frac{4-x}{x(x^2+4x+2)^2} = \frac{1}{x} + \frac{-(x+4)}{x^2+4x+2} + \frac{-(2x+9)}{(x^2+4x+2)^2}$$

Example: Decompose into partial fractions and integrate $\frac{4-x}{x^5+4x^3+4x}$.

We first factor the denominator: $x^5+4x^3+4x = x(x^2+2)^2$.

Then we write the decomposition: $\frac{4-x}{x^5+4x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$.

Find the coefficients: $A=1$, $B=-1$, $C=0$, $D=-2$, $E=-1$.

We can then integrate:

$$\int \frac{4-x}{x^5+4x^3+4x}\,dx = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+2} - \int \frac{2x+1}{(x^2+2)^2}\,dx$$

As:

$$\int \frac{dx}{x} = \ln|x| + C_1$$

$$\int \frac{x\,dx}{x^2+2} = \cdots = \frac{1}{2}\ln|x^2+2| + C_1$$

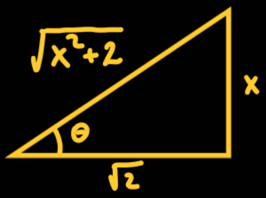
$u = x^2+2$
$du = 2x\,dx$

$$\int \frac{2x+1}{(x^2+2)^2}\,dx = \int \frac{2x\,dx}{(x^2+2)^2} + \int \frac{dx}{(x^2+2)^2}$$

$$\int \frac{2x\,dx}{(x^2+2)^2} = \cdots = \frac{-1}{x^2+2} + C_1$$

$u = x^2+2$
$du = 2x\,dx$

$$\int \frac{dx}{(x^2+2)^2} = \int \frac{\sqrt{2}\cdot \sec^2(\theta)\, d\theta}{(2\cdot \tan^2(\theta)+2)^2} = \int \frac{\sqrt{2}\cdot \sec^2(\theta)\, d\theta}{4\cdot \sec^4(\theta)} = \frac{\sqrt{2}}{4}\int \cos^2(\theta)\, d\theta =$$



$x = \sqrt{2}\cdot \tan(\theta)$ $\qquad$ $dx = \sqrt{2}\cdot \sec^2(\theta)\, d\theta$

$x^2+2 = 2\cdot \tan^2(\theta)+2 = 2\cdot \sec^2(\theta)$

$\tan^2(\theta)+1 = \sec^2(\theta)$

$$= \frac{\sqrt{2}}{4}\left(\frac{\theta}{2} + \frac{\sin(\theta)\cdot \cos(\theta)}{2}\right) + c =$$

integration by parts

$$= \frac{\sqrt{2}}{8}\cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8}\cdot \frac{x}{\sqrt{x^2+2}}\cdot \frac{\sqrt{2}}{\sqrt{x^2+2}} + c =$$

$$= \frac{1}{4\sqrt{2}}\cdot \arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{4(x^2+2)} + c$$

We finally have:

$$\int \frac{4-x}{x^5+4x^3+4x}\, dx = \ln|x| - \frac{1}{2}\ln|x^2+2| + \frac{\frac{x}{4}-1}{x^2+2} - \frac{1}{4\sqrt{2}}\cdot \arctan\left(\frac{x}{\sqrt{2}}\right).$$

If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, do the long division of polynomials.

Section 9.1.: Arc length and surface area.

The length of a curve is called arc length.

Formula for arc length:

Assume that $f'(x)$ exists and is continuous on $[a,b]$.

Then the arc length of $f(x)$ over $[a,b]$ is:

$$s = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Example: Find the arc length of $f(x) = \frac{x^3}{12} + x^{-1}$ over $[1,3]$.

Compute first:

$$1+(f'(x))^2 = 1+\left(\frac{1}{4}x^2 - x^{-2}\right)^2 = 1+\frac{x^4}{16} - \frac{1}{2} + x^{-4} = \frac{x^4}{16} + \frac{1}{2} + x^{-4} = \left(\frac{x^2}{4} + x^{-2}\right)^2.$$

Compute the arc length:

$$s = \int_1^3 \sqrt{1+(f'(x))^2}\,dx = \int_1^3 \left(\frac{x^2}{4} + x^{-2}\right)dx = \frac{x^3}{12} - x^{-1}\Big|_1^3 = \left(\frac{9}{4} - \frac{1}{3}\right) - \left(\frac{1}{12} - 1\right) = \frac{17}{6}.$$

Example: Find the arc length of $f(x) = \cosh(x)$ over $[0,a]$.

Compute first:

$$1+(f'(x))^2 = 1+(\sinh(x))^2 = \cosh^2(x)$$

$\frac{d}{dx}(\cosh(x)) = \sinh(x)$ $\qquad$ $\cosh^2(x) + \sinh^2(x) = 1$

Since $\cosh^2(x) \geq 0$ for $x \geq 0$ we have $\sqrt{\cosh^2(x)} = \cosh(x)$, we compute the arc length:

$$s = \int_0^a \sqrt{1+(f'(x))^2}\,dx = \int_0^a \sqrt{\cosh^2(x)}\,dx = \int_0^a \cosh(x) = \sinh(x)\Big|_0^a = \sinh(a).$$

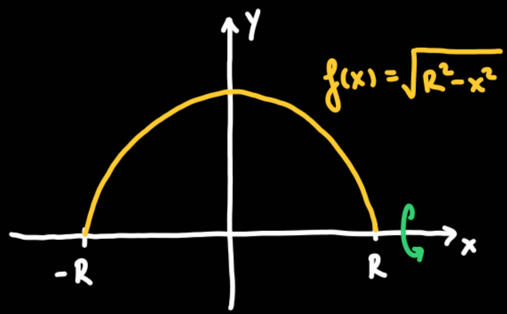
Formula for the area of a surface of revolution:

Assume that $f(x) \geq 0$ and that $f'(x)$ exists and is continuous on $[a,b]$. The surface area S of the surface obtained by rotating the graph of $f(x)$ about the x-axis for

$a \le x \le b$ is: $\quad S = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$

Example: Calculate the surface area of a sphere of radius R.

A semicircle of radius R is given by the function $f(x) = \sqrt{R^2-x^2}$. We obtain a sphere by rotating about the x-axis.



Compute first:

$$f'(x) = \frac{-x}{\sqrt{R^2-x^2}}, \qquad 1+(f'(x))^2 = 1+\frac{x^2}{R^2-x^2} = \frac{R^2}{R^2-x^2}.$$

Compute the surface area:

$$S = 2\pi\int_{-R}^{R} f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\cdot\frac{R}{\sqrt{R^2-x^2}}\,dx = 2\pi R\int_{-R}^{R} dx =$$

$$= 2\pi(R-(-R)) = 4\pi R^2.$$

Example: Find the surface area of the surface obtained by rotating the graph of $f(x) = \sqrt{x} - \frac{\sqrt[3]{x^2}}{3}$ about the x-axis for $1 \le x \le 3$.

Compute first:

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}, \qquad 1+(f'(x))^2 = 1+\left(\frac{1}{2\sqrt{x}}-\frac{\sqrt{x}}{2}\right)^2 = 1+\frac{x^{-1}-2+x}{4} =$$

$$= \frac{x^{-1}+2+x}{4} = \frac{\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)^2}{}$$

$$= \frac{\quad}{4} = \left(\frac{\quad}{2}\right)$$

Compute the surface area:

$$S = 2\pi \int_1^3 f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi \int_1^3 \left(x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3}\right)\left(\frac{x^{\frac{1}{2}} + x^{\frac{-1}{2}}}{2}\right)^2 dx =$$

$$= \pi \int_1^3 \left(1 + \frac{2x}{3} - \frac{x^2}{3}\right) dx = \pi\left(x + \frac{x^2}{3} - \frac{x^3}{9}\right)\Big|_1^3 = \frac{16\pi}{9}.$$

Section 9.4.: Taylor polynomials.

For this section we assume that $f(x)$ is defined on some open interval and that all higher derivatives $f^{(k)}(x)$ exist. Let I be an interval and x, a points in I.

Taylor polynomial centered at $x=a$:

$$\boxed{T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j}$$

When $a=0$, we also call this the $\boxed{\text{Maclaurin polynomial.}}$

Example: Compute the third and fourth Maclaurin polynomials for $f(x) = e^x$.

Since $f^{(k)}(x) = e^x$ for all positive integer k, we have at $x=0$:

$$f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = 1.$$

The third Maclaurin polynomial is:

$$T_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

We obtain $T_4(x)$ by adding the term of degree four to $T_3(x)$:

$$T_4(x) = T_3(x) + \frac{f^{(4)}(0)}{24}(x-0)^4 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

Example: Find the Taylor polynomials of $f(x) = \ln(x)$ centered at $a = 1$.

Compute the derivatives:

$$f(x) = \ln(x), \quad f'(x) = x^{-1}, \quad f''(x) = -x^{-2}, \quad f'''(x) = 2x^{-3}, \quad f^{(4)}(x) = -3\cdot 2\cdot x^{-4},$$

so the pattern is $f^{(k)}(x) = \underbrace{(-1)^{k-1}}_{\text{alternating sign}} \cdot \underbrace{(k-1)!}_{\text{coefficients from the exponent rule}}\, x^{-k}$ for $k \geq 1$. Then:

$$\frac{f^{(k)}(1)}{k!}(x-1)^k = \frac{(-1)^{k-1}\cdot(k-1)!\cdot 1^k}{k!}(x-1)^k = \frac{(-1)^{k-1}}{k}(x-1)^k \quad \text{so:}$$

$$T_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^n\cdot(x-1)^n}{n} = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j}(x-1)^j.$$

Useful Maclaurin and Taylor polynomials:

$f(x)$	a	polynomial
e^x	0	$T_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!}$
$\sin(x)$	0	$T_{2n+1}(x) = T_{2n+2}(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2j+1}}{(2j+1)!}$
$\cos(x)$	0	$T_{2n}(x) = T_{2n+1}(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2j}}{(2j)!}$
$\ln(x)$	1	$T_n(x) = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j}\cdot(x-1)^j$
$\frac{1}{1-x}$	0	$T_n(x) = \sum_{j=0}^{n} x^j$

Error bound: Assume that $f^{(n+1)}(x)$ exists and is continuous, let K be a number such that $|f^{(n+1)}(u)| \leq K$ for all u between a and x, and let $T_n(x)$ be the n-th Taylor polynomial centered at $x=a$. Then:

$$\boxed{|f(x) - T_n(x)| \leq K \cdot \frac{|x-a|^{n+1}}{(n+1)!}}$$

Namely, the error of approximating $f(x)$ by $T_n(x)$ is proportional to $\frac{|x-a|^{n+1}}{(n+1)!}$.

Example: Let $f(x) = \ln(x)$ and $a=1$. Bound the error of $T_3(x)$ at $x=1.2$.

Step 1: Find a value of K. We want $|f^{(4)}(u)| \leq K$ for all u between $a=1$ and $x=1.2$. Since $f^{(4)}(x) = -6x^{-4}$, and $|f^{(4)}(x)| = 6x^{-4}$ is decreasing for $x>0$, its maximum value on $[1, 1.2]$ is $|f^{(4)}(1)| = 6$. Take $K=6$.

Step 2: Apply the error bound.

$$|\ln(1.2) - T_3(1.2)| \leq 6 \cdot \frac{|1.2-1|^4}{4!} = \frac{0.2^4}{4} = \frac{\left(\frac{2}{10}\right)^4}{4} = \frac{16}{40000} = \frac{1}{2500} = 0.0004.$$

Example: Let $f(x) = \cos(x)$ and $a=0.2$. Find an integer n such that the n-th Maclaurin polynomial $T_n(x)$ has an error of less than 10^{-5}.

Step 1: Find a value of K. Since $|f^{(n)}(x)| = |\cos(x)|$ for n even, and $|f^{(n)}(x)|$ is $|\sin(x)|$ for n odd, we always have $|f^{(n)}(x)| \leq 1$. Take $K=1$.

Step 2: Use the error bound to find a value of n.

$$|\cos(0.2) - T_n(0.2)| \leq 1 \cdot \frac{|0.2-0|^{n+1}}{(n+1)!} = \frac{0.2^{n+1}}{(n+1)!} < 10^{-5} = \frac{1}{100000}$$

we must choose n so that this happens.

This is not solvable, so we find n by checking several values:

n	2	3	4
$\frac{0.2^{n+1}}{(n+1)!}$	$\frac{1}{750} \approx 0.0013$	$\frac{1}{15000} \approx 0.00007$	$\frac{1}{375000} \approx 0.0000027$

We have that the error is less than 10^{-5} for $n=4$.

Section 8.6.: Improper integrals

Improper integrals represent areas of unbounded regions. For a region to be unbounded, the interval of integration may be infinite or the function that is being integrated may tend to infinity.

Improper integral: The improper integral of $f(x)$ over $[a,\infty)$ is, if it exists:

$$\boxed{\int_a^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.}$$

We say that the integral converges if the limit exists, and that it diverges if the limit does not exist. The improper integral of $f(x)$ over $(-\infty, a]$ is, if it exists:

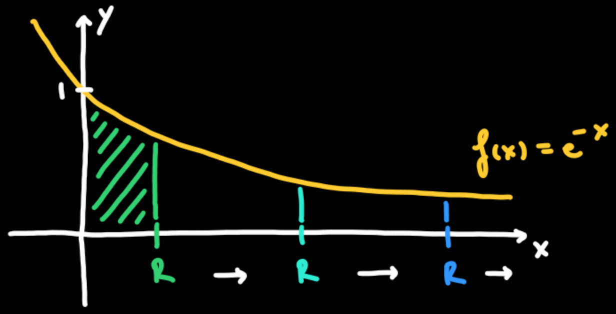
$$\boxed{\int_{-\infty}^{a} f(x)\,dx = \lim_{R\to-\infty} \int_R^a f(x)\,dx.}$$

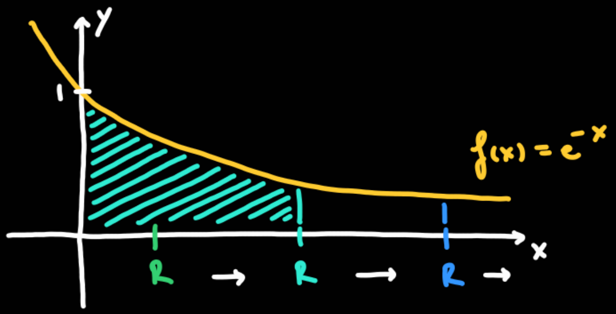
An integral over $(-\infty,\infty)$ is computed by separating the infinities:

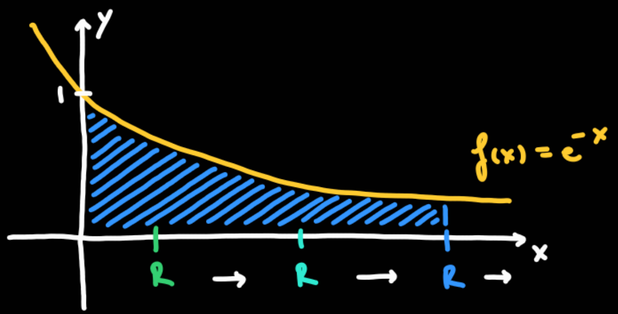
$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{-\infty} f(x)\,dx}$$

Example: Evaluate:

$$\int_0^{\infty} e^{-x}\,dx = \lim_{R\to\infty}\int_0^R e^{-x}\,dx = \lim_{R\to\infty}\left(-e^{-x}\right)\Big|_0^R = \lim_{R\to\infty}\left(1-e^{-R}\right) = 1.$$







Example: Evaluate:

$$\int_2^{\infty}\frac{dx}{x^3} = \lim_{R\to\infty}\int_2^R \frac{dx}{x^3} = \lim_{R\to\infty}\left(\frac{-x^{-2}}{2}\right)\Big|_2^R = \lim_{R\to\infty}\left(\frac{1}{8}-\frac{1}{2R^2}\right) = \frac{1}{8}.$$

Example: Determine whether $\int_{-\infty}^{-1}\frac{dx}{x}$ converges.

$$\int_{-\infty}^{-1}\frac{dx}{x} = \lim_{R\to-\infty}\int_R^{-1}\frac{dx}{x} = \lim_{R\to-\infty} \ln|x|\Big|_R^{-1} = \lim_{R\to-\infty}\left(\ln(1)-\ln|R|\right) = -\infty$$

The improper integral diverges.

The p-integral over $[a,\infty)$:

$$\boxed{\int_a^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{p-1} & \text{if } p>1, \\ \text{diverges} & \text{if } p\le 1. \end{cases}}$$

__Example:__ Evaluate:

$$\int_0^\infty xe^{-x}dx = \lim_{R\to\infty}\int_0^R x\cdot e^{-x}dx = \lim_{R\to\infty} -(x+1)e^{-x}\Big|_0^R = \lim_{R\to\infty}\left(-(R+1)e^{-R}+1\right) = \lim_{R\to\infty}\left(1-\frac{R+1}{e^R}\right) =$$

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} = -(x+1)e^{-x} + C.$$

$u = x \quad du = 1$

$dv = e^{-x} \quad v = -e^{-x}$

$$= 1 - \lim_{R\to\infty}\frac{R+1}{e^R} = 1 - \lim_{R\to\infty}\frac{1}{e^R} = 1.$$

L'Hôpitals Rule

__Integrands with infinite discontinuities:__

If $f(x)$ is continuous on $[a,b)$ but discontinuous at $x=b$, then:

$$\boxed{\int_a^b f(x)\,dx = \lim_{R\to b^-}\int_a^R f(x)\,dx.}$$

If $f(x)$ is continuous on $(a,b]$ but discontinuous at $x=a$, then:

$$\boxed{\int_a^b f(x)\,dx = \lim_{R\to a^+}\int_R^b f(x)\,dx.}$$

We say that the integral converges if the limit exists, and that it diverges if the limit does __not__ exist.

__Example:__ Evaluate:

$$\int_0^9 \frac{dx}{\sqrt{x}} = \lim_{R\to 0^+}\int_R^9 x^{-\frac{1}{2}}dx = \lim_{R\to 0^+} 2x^{1/2}\Big|_R^9 = \lim_{R\to 0^+}(6-2\sqrt{R}) = 6.$$

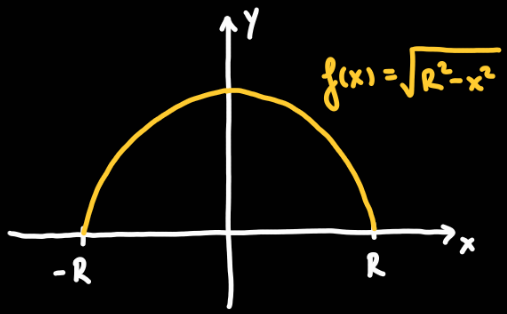
$$\int_0^{\frac{1}{2}} \frac{dx}{x} = \lim_{R\to 0^+} \int_R^{\frac{1}{2}} \frac{dx}{x} = \lim_{R\to 0^+} \ln|x| \Big|_R^{\frac{1}{2}} = \lim_{R\to 0^+} \left(\ln\left(\tfrac{1}{2}\right) - \ln(R)\right) = +\infty .$$

The p-integral over $[0,a]$:

$$\boxed{\int_0^a \frac{dx}{x^p} = \begin{cases} \frac{a^{1-p}}{1-p} & \text{if } p<1, \\ \text{diverges} & \text{if } p \geq 1. \end{cases}}$$

Example: Calculate the length of one quarter of the unit circle.

A semicircle of radius R is given by the function $f(x) = \sqrt{R^2 - x^2}$. We obtain a unit circle by setting $R=1$. One quarter is given by the interval $[0,1]$.



Compute first:

$$f'(x) = \frac{-x}{\sqrt{1-x^2}}, \quad 1 + (f'(x))^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2} .$$

Compute the arc length:

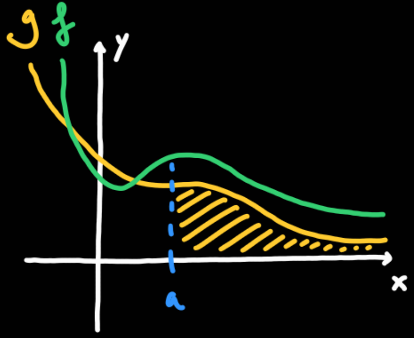
$$s = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{R\to 1^-} \int_0^R \frac{dx}{\sqrt{1-x^2}} = \lim_{R\to 1^-} \arcsin(x) \Big|_0^R =$$

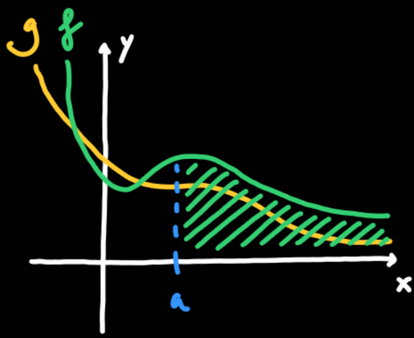
$$= \lim_{R\to 1^-} \left(\arcsin(R) - \arcsin(0)\right) = \arcsin(1) - \arcsin(0) = \frac{\pi}{2} .$$

Comparison test for improper integrals: Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$.

If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.





Example: Show that $\int_1^\infty \frac{e^{-x}}{x}\,dx$ converges.

For $x \geq 1$ we have $0 \leq \frac{e^{-x}}{x} \leq e^{-x}$, so $0 \leq \int_1^\infty \frac{e^{-x}}{x}\,dx \leq \int_1^\infty e^{-x}\,dx = \frac{1}{e}$.

$$\int_1^\infty e^{-x}\,dx = \lim_{R\to\infty} \int_1^R e^{-x}\,dx = \lim_{R\to\infty} -e^{-x}\Big|_0^R = \frac{1}{e}$$

Example: Show that $\int_1^\infty \frac{dx}{\sqrt{x}+e^{3x}}$ converges.

For $x \geq 1$ we have $\sqrt{x}+e^{3x} \geq e^{3x}$ so $\frac{1}{\sqrt{x}+e^{3x}} \leq \frac{1}{e^{3x}}$. Moreover:

$$\int_1^\infty e^{-3x}\,dx = \lim_{R\to\infty} \int_1^R e^{-3x}\,dx = \lim_{R\to\infty} \frac{-e^{-3x}}{3}\Big|_0^R = \frac{e^{-3}}{3} \quad \text{converges.}$$

By the Comparison Test, $\int_1^\infty \frac{dx}{\sqrt{x}+e^{3x}}$ converges.

However, we also have $\sqrt{x}+e^{3x} \geq \sqrt{x}$ so $\frac{1}{\sqrt{x}+e^{3x}} \leq \frac{1}{\sqrt{x}}$ for $x \geq 1$. This does <u>not</u> help because $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges, so it says nothing about our smaller integral.

Section 11.1: Sequences

A sequence is an ordered collection of numbers defined by a function $f(n)$, n integer.

The values $a_n = f(n)$ are called the terms of the sequence, n is called the index. When a_n is given by a formula, we refer to it as the general term.

<u>Example:</u> 1) $a_n = 1 - \frac{1}{n}$, $n \geq 1$: $\quad 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

2) $a_n = (-1)^n n$, $n \geq 0$: $\quad 0, -1, 2, -3, 4, \dots$

<u>Example:</u> Compute a_2, a_3, a_4 for the sequence defined recursively by $a_1 = 1$, $a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$.

$$a_2 = \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}$$

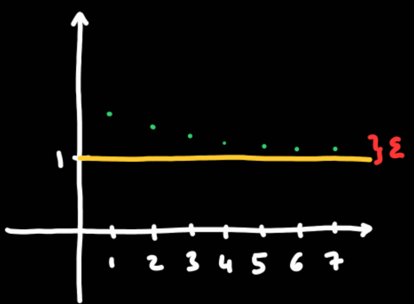
$$a_3 = \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\frac{3}{2}}\right) = \frac{17}{12}$$

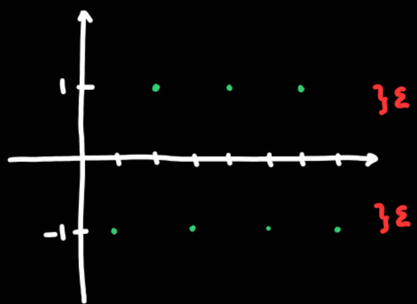
$$a_4 = \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right) = \frac{1}{2}\left(\frac{17}{2} + \frac{2}{\frac{17}{2}}\right) = \frac{577}{408}$$

<u>Limit of a sequence:</u>

We say that $\{a_n\}$ converges to a limit L, and we write $\lim_{n \to \infty} a_n = L$, if for every $\varepsilon > 0$ there is a number M such that $|a_n - L| < \varepsilon$ for all $n > M$.

If no limit exists, we say that $\{a_n\}$ diverges. If the terms increase without bound, we say that $\{a_n\}$ diverges to infinity.





$a_n = \frac{n+4}{n+1}$ $\qquad\qquad\qquad$ $a_n = (-1)^n$

Example: Let $a_n = \frac{n+4}{n+1}$. Prove that $\lim\limits_{n\to\infty} a_n = 1$.

We have $|a_n - 1| = \left|\frac{n+4}{n+1} - 1\right| = \frac{3}{n+1}$. Thus $|a_n - 1| < \varepsilon$ if $\frac{3}{n+1} < \varepsilon$, that is $n > \frac{3}{\varepsilon} - 1$.

Set $M = \frac{3}{\varepsilon} - 1$, now for $n > M$ we have:

$$|a_n - 1| = \left|\frac{n+4}{n+1} - 1\right| = \frac{3}{n+1} < 3 \cdot \frac{1}{\frac{3}{\varepsilon} - 1 + 1} = 3 \cdot \frac{1}{\frac{3}{\varepsilon}} = \varepsilon.$$ Hence $\lim\limits_{n\to\infty} a_n = 1$.

Sequence defined by a function:

If $\lim\limits_{x\to\infty} f(x)$ exists, then the sequence $a_n = f(n)$ converges to the same limit.

$$\boxed{\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x).}$$

Example: Find the limit of the sequence with general term $a_n = \frac{n^2-2}{n^2}$.

Note that $a_n = \frac{n^2-2}{n^2} = 1 - \frac{2}{n^2}$, consider $f(x) = 1 - \frac{2}{x^2}$, now:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(1 - \frac{2}{x^2}\right) = 1.$$

Example: Calculate:

$$\lim_{n\to\infty} \frac{n + \ln(n)}{n^2} = \lim_{x\to\infty} \frac{x + \ln(x)}{x^2} \underset{\text{L'Hôpitals Rule}}{=} \lim_{x\to\infty} \frac{1 + \frac{1}{x}}{2x} = 0.$$

Example: Compute for $r \geq 0$ and $c > 0$:

$$\lim c \cdot r^n = \lim c \cdot r^x = c \cdot \lim r^x = \begin{cases} 0 & \text{if } 0 \leq r < 1. \\ c & \text{if } r = 1 \end{cases}$$

$n \to \infty$ $\quad x \to \infty$ $\quad x \to \infty$ $\quad \infty$ if $r > 1$.

Limit laws for sequences:

Assume that $\{a_n\}$ and $\{b_n\}$ are convergent sequences with $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$.

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M.$$

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M} \text{ if } M \neq 0.$$

$$\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n = c \cdot L \text{ for any constant } c.$$

Squeeze theorem for sequences:

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that for some number M,

$b_n \le a_n \le c_n$ for $n > M$ and $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} c_n$. Then $\lim_{n\to\infty} a_n = L$.

Example: Show that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

We have $-|a_n| \le a_n \le |a_n|$, and we have $\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = 0$, so by the Squeeze theorem we have $\lim_{n\to\infty} a_n = 0$.

Example: Compute for $r < 0$ and $c \neq 0$:

$$\lim_{n\to\infty} c \cdot r^n = \begin{cases} 0 & \text{if } -1 < r < 0. \end{cases}$$

$\begin{cases} \quad \\ \text{diverges if } r \le -1. \end{cases}$

If $-1<r<0$, then $0<|r|<1$ and $\lim_{n\to\infty} |cr^n| = \lim_{n\to\infty} |c|\cdot|r^n| = 0$. Thus $\lim_{n\to\infty} cr^n = 0$ by the Squeeze theorem.

If $r=-1$ then the sequence $a_n = cr^n = (-1)^n c$ alternates in sign and does <u>not</u> approach a limit.

If $r<-1$ then $a_n = cr^n$ alternates in sign and $|a_n| = |cr^n|$ grows arbitrarily large and does <u>not</u> approach a limit.

<u>Example</u>: Prove that $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for all R.

Assume that $R>0$ and let M be the positive integer such that $M \le R < M+1$. For $n>M$ we write:
$$\frac{R^n}{n!} = \underbrace{\left(\frac{R}{1}\cdot\frac{R}{2}\cdots\frac{R}{M}\right)}_{C}\cdot \underbrace{\frac{R}{M+1}}_{\le 1}\cdot\underbrace{\frac{R}{M+2}}_{\le 1}\cdots\underbrace{\frac{R}{n-1}}_{\le 1}\cdot\frac{R}{n} \le C\cdot\frac{R}{n}. \text{ Thus:}$$

$0 \le \frac{R^n}{n!} \le C\cdot\frac{R}{n}$, and since $\lim_{n\to\infty} C\cdot\frac{R}{n} = 0$, by the Squeeze theorem we have $\lim_{n\to\infty}\frac{R^n}{n!} = 0$ as claimed.

If $R<0$, we have that $\left|\frac{R^n}{n!}\right|$ tends to zero, so $\lim_{n\to\infty}\frac{R^n}{n!} = 0$.

If $R=0$, then $\lim_{n\to\infty}\frac{R^n}{n!} = \lim_{n\to\infty} 0 = 0$.

We can bring a limit inside a continuous function.

$$\boxed{\text{If } f(x) \text{ is continuous and } \lim_{n\to\infty} a_n = L, \text{ then } \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).}$$

Example: Compute $\lim_{n\to\infty} f(a_n)$ for $a_n = \frac{3n}{n+1}$ and $f(x) = x^2$.

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty}\left(\frac{3n}{n+1}\right)^2 = f\left(\lim_{n\to\infty}\frac{3n}{n+1}\right) = \left(\lim_{n\to\infty}\frac{3n}{n+1}\right)^2 = 3^2 = 9.$$

Bounded sequences: A sequence $\{a_n\}$ is:

> Bounded from above if there is a number M such that $a_n \leq M$ for all n. The number M is called an upper bound.
>
> Bounded from below if there is a number m such that $a_n \geq m$ for all n. The number m is called a lower bound.

Convergent sequences are bounded:

> If $\{a_n\}$ converges, then $\{a_n\}$ is bounded.

Bounded monotonic sequences converge:

> If $\{a_n\}$ is increasing and $a_n \leq M$, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \leq M$.
>
> If $\{a_n\}$ is decreasing and $a_n \geq m$, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \geq m$.

Example: Does $\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right)$ exist?

We have $a_n = \sqrt{n+1} - \sqrt{n}$, so consider $f(x) = \sqrt{x+1} - \sqrt{x}$. This function is decreasing because its derivative is negative: $f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0$ for $x > 0$. Then $a_n = f(n)$ is decreasing.

Moreover $a_n > 0$ for all n, so $m = 0$ is a lower bound for the sequence. Since a bounded monotonic sequence converges, $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$ exists. Note:

$$\sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}, \text{ and now:}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \lim_{x\to\infty} \frac{1}{\sqrt{x+1}+\sqrt{x}} = 0.$$

Section 11.2.: Summing an infinite series.

An infinite series is an infinite sum: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$. We compute the sum of an infinite series as the limit of its partial sums.

Infinite series: $a_1 + a_2 + a_3 + a_4 + \cdots$

N-th partial sum: $S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n$.

If the series begins at k, then $S_N = \sum_{n=k}^{N} a_n$.

Convergence of an infinite series:

An infinite series $\sum_{n=k}^{\infty} a_n$ converges to the sum S if its partial sums converge to S:

$\lim_{N\to\infty} S_N = S$. In this case, we write $S = \sum_{n=k}^{\infty} a_n$.

If the limit does not exist, we say that the infinite series diverges.

If the limit is infinite, we say that the infinite series diverges to infinity.

Example: Telescopic series. Compute $S=\sum_{n=1}\frac{1}{n(n+1)}=1.$

We compute the partial sums using that $\frac{1}{n(n+1)}=\frac{1}{n}+\frac{1}{n+1}$:

$$S_1=\frac{1}{1\cdot 2}=1-\frac{1}{2}$$

$$S_2=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)=1-\frac{1}{3}.$$

$$S_3=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)=1-\frac{1}{4}\text{, and in general:}$$

$$S_N=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\dots+\left(\frac{1}{N}+\frac{1}{N+1}\right)=1-\frac{1}{N+1}.$$

Now, the sum S is the limit of the partial sums:

$$S=\lim_{N\to\infty}S_N=\lim_{N\to\infty}\left(1-\frac{1}{N}\right)=1.$$

> It is important to keep in mind the difference between a sequence $\{a_n\}$ and an infinite series $\sum_{n=1}^{\infty}a_n$.

Linearity of infinite series:

> If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n\pm b_n)$ and $\sum c a_n$ also converge (for c any constant), and:
>
> $$\sum a_n+\sum b_n=\sum(a_n+b_n),\quad \sum a_n-\sum b_n=\sum(a_n-b_n),\quad \sum c a_n=c\sum a_n.$$

Sum of a geometric series: Let $c\neq 0$. If $|r|<1$ then:

> $$\sum_{n=0}^{\infty}cr^n=c+cr+cr^2+\dots+=\frac{c}{1-r}\quad\text{and}\quad\sum_{n=M}^{\infty}cr^n=cr^M+cr^{M+1}+cr^{M+2}+\dots=\frac{cr^M}{1-r}.$$

If $|r| \geq 1$ then the geometric series diverges.

Example: Compute:

$$\sum_{n=0}^{\infty} \frac{2+3^n}{5^n} = \sum_{n=0}^{\infty} \frac{2}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^n} = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{5^n} + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = 2 \cdot \frac{1}{1-\frac{1}{5}} + \frac{1}{1-\frac{3}{5}} = 2 \cdot \frac{5}{4} + \frac{5}{2} = 5.$$

Divergence test:

$$\boxed{\text{If the } n\text{-th term } a_n \text{ does not converge to zero, then the series } \sum_{n=1}^{\infty} a_n \text{ diverges.}}$$

Example: Prove the divergence of $\sum_{n=1}^{\infty} \frac{n}{4n+1}$.

We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{4n+1} = \lim_{n\to\infty} \frac{1}{4+\frac{1}{n}} = \lim_{x\to\infty} \frac{1}{4+\frac{1}{x}} = \frac{1}{4}$. Since the general term does not converge, the series diverge.

Example: Prove the divergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Note that the general term $a_n = \frac{1}{\sqrt{n}}$ tends to zero. However:

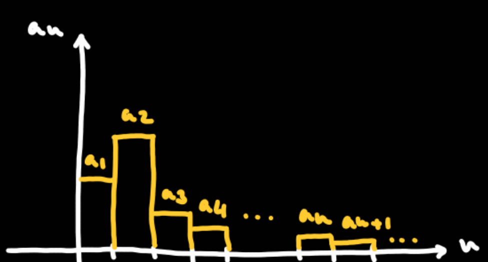
$$S_N = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{N}} \geq \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \cdots + \frac{1}{\sqrt{N}} = N \cdot \frac{1}{\sqrt{N}} = \sqrt{N}$$

$\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{N}}$ for $n \leq N$.

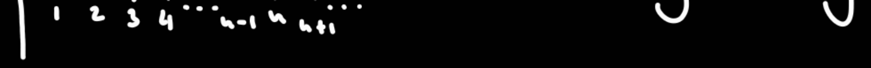
So $\lim_{N\to\infty} S_N \geq \lim_{N\to\infty} \sqrt{N} = \infty$, and the series diverges.

Section 11.3.: Convergence of series with positive terms.

We consider $\boxed{\text{positive series}}$, namely series $\sum a_n$ where $a_n > 0$ for all n.



We can visualize the terms of a positive series as rectangles of height a_n and width 1.



The partial sum $S_N = a_1 + \cdots + a_N$ is the area of the first N rectangles. Since $a_{N+1} > 0$ then

$S_N < S_{N+1}$ so the partial sums form an increasing sequence.

<u>Dichotomy for positive series:</u>

> For $S = \sum a_n$ a positive series we have either:
>
> (i) The partial sums S_N are bounded above and S converges.
>
> (ii) The partial sums S_N are <u>not</u> bounded above and S diverges.

<u>Integral test:</u>

> Let $a_n = f(n)$ where $f(x)$ is positive, decreasing, and continuous for $x \geq 1$. Then:
>
> (i) If $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
>
> (ii) If $\int_1^\infty f(x)\,dx$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Example:</u> The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges: let $f(x) = \frac{1}{x}$, then $a_n = \frac{1}{n} = f(n)$, $f(x)$ is

positive, decreasing, and continuous for $x \geq 1$. Then since:

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{dx}{x} = \lim_{R\to\infty} \ln(R) = \infty$$

the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the integral test.

<u>Example:</u> Does $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converge or diverge?

The function $f(x) = \frac{x}{(x^2+1)^2}$ is positive and continuous for $x \geq 1$. Moreover:

$f'(x) = \frac{1-3x^2}{(x^2+1)^3} < 0$ for $x \geq 1$, so $f(x)$ is decreasing. Then since:

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{x\,dx}{(x^2+1)^2} = \lim_{R\to\infty} \int_1^R \frac{x\,dx}{(x^2+1)^2} = \lim_{R\to\infty} \frac{1}{2}\int_2^R \frac{du}{u^2} = \lim_{R\to\infty} \left.\frac{-1}{2u}\right|_2^R =$$

$u = x^2+1$
$du = 2x\,dx$

$= \lim_{R\to\infty}\left(\frac{1}{4} - \frac{1}{2R}\right) = \frac{1}{4}$, by the integral test the series $\sum_{n=1}^\infty \frac{n}{(n^2+1)^2}$ converges.

Convergence of p-series:

> The infinite series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p>1$ and diverges if $p \leq 1$.

If $p \leq 0$ then the general term $a_n = n^{-p}$ does not tend to zero, so the series diverges.

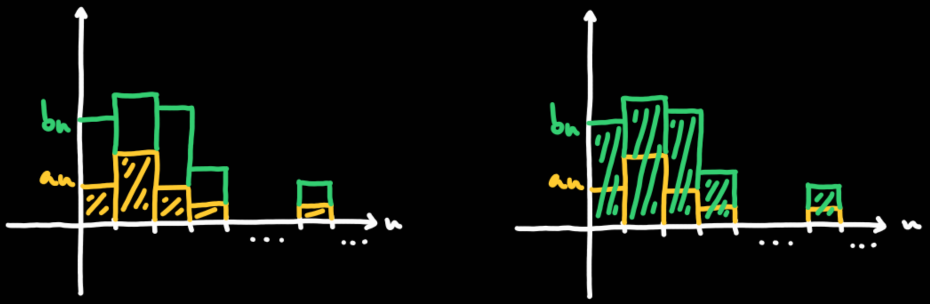
If $p>0$ then $f(x) = \frac{1}{x^p}$ is positive, decreasing, and continuous for $x \geq 1$. Since:

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p \leq 1 \end{cases}$$

then by the integral test $\sum_{n=1}^\infty \frac{1}{n^p}$ converges for $p>1$ and diverges for $p \leq 1$.

Comparison test:

> Assume there exists $M>0$ such that $0 \leq a_n \leq b_n$ for all $n>M$.
>
> (i) If $\sum_{n=1}^\infty b_n$ converges then $\sum_{n=1}^\infty a_n$ converges.
>
> (ii) If $\sum_{n=1}^\infty a_n$ diverges then $\sum_{n=1}^\infty b_n$ diverges.



<u>Example:</u> Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$ converge or diverge?

For $n \geq 1$ we have $\frac{1}{\sqrt{n}\,3^n} \leq \frac{1}{3^n}$. Moreover $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges because it is a geometric series with $r = \frac{1}{3} < 1$. By the Comparison Test, the smaller series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$ converges.

<u>Example:</u> Does $\sum_{n=2}^{\infty} \frac{1}{(n^2+3)^{\frac{1}{3}}}$ converge or diverge?

We observe that $\frac{1}{(n^2+3)^{\frac{1}{3}}}$ behaves like $\frac{1}{n^{2/3}}$ when n goes to infinity. Since $\frac{1}{n} \leq \frac{1}{n^{2/3}}$, we would like to prove that $\frac{1}{n} \leq \frac{1}{(n^2+3)^{\frac{1}{3}}}$. This inequality is equivalent to $n^2+3 \leq n^3$. Consider $f(x) = x^3 - (x^2-3)$, which is positive for $x \geq 2$. Since:

$f'(x) = 3x^2 - 2x = 3x(x-\frac{2}{3}) > 0$ for $x \geq 2$, then $f(x)$ is increasing. Moreover $f(2) = 1$,

so $f(x) \geq 1 \geq 0$ for all $x \geq 2$, so $x^3 - (x^2-3) \geq 0$ for all $x \geq 2$, so $n^3 - (n^2-3) \geq 0$ for all $n \geq 2$, so $\frac{1}{n} \leq \frac{1}{(n^2+3)^{\frac{1}{3}}}$ for all $n \geq 2$. Now $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so by the Comparison test $\sum_{n=2}^{\infty} \frac{1}{(n^2+3)^{\frac{1}{3}}}$ also diverges.

<u>Example:</u> Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converge or diverge?

We have $\frac{1}{n(\ln(n))^2} \leq \frac{1}{n}$ for $n \geq 3$, so we can compare $\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^2}$ with $\sum_{n=3}^{\infty} \frac{1}{n}$.

However the harmonic series diverges, and the Comparison test does not give information about the smaller series. Using the integral test, we have:

$$\int_2^\infty \frac{dx}{x(\ln(x))^2} = \int_{\ln(2)}^\infty \frac{du}{u^2} = \lim_{R\to\infty}\int_{\ln(2)}^R \frac{du}{u^2} = \lim_{R\to\infty} \frac{-1}{u}\Big|_{\ln(2)}^R = \lim_{R\to\infty}\left(\frac{1}{\ln(2)} - \frac{1}{R}\right) = \frac{1}{\ln(2)}$$

$u = \ln(x),\ du = \frac{1}{x}dx$

and the infinite series converge.

Limit comparison test:

> Let $\{a_n\}$ and $\{b_n\}$ be positive sequences such that $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists.
>
> If $L > 0$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.
>
> If $L = \infty$ and $\sum a_n$ converges then $\sum b_n$ converges.
>
> If $L = 0$ and $\sum b_n$ converges then $\sum a_n$ converges.

Example: Does $\sum_{n=2}^\infty \frac{n^2}{n^4-n-1}$ converge or diverge?

Let $a_n = \frac{n^2}{n^4-n-1}$ and $b_n = \frac{1}{n^2}$, now:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^4-n-1}\cdot\frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{1-n^{-3}-n^{-4}} = 1.$$

Since $\sum_{n=2}^\infty \frac{1}{n^2}$ is a converging p-series, then $\sum_{n=2}^\infty \frac{n^2}{n^4-n-1}$ also converges by limit comparison.

Example: Does $\sum_{n=3}^\infty \frac{1}{\sqrt{n^2+4}}$ converge or diverge?

Let $a_n = \frac{1}{\sqrt{n^2+4}}$ and $b_n = \frac{1}{n}$, now:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+4}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{4}{n^2}}} = 1.$$

Since $\sum_{n=3}^{\infty} \frac{1}{n}$ is a diverging p-series, then $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+4}}$ also diverges by the limit comparison test.

Section 11.4.: Absolute and conditional convergence

Absolute convergence: $\boxed{\text{The series } \sum a_n \text{ converges absolutely if } \sum |a_n| \text{ converges.}}$

Example: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges absolutely because $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

Absolute convergence implies convergence: $\boxed{\text{If } \sum |a_n| \text{ converges then } \sum a_n \text{ converges.}}$

Example: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges because $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right|$ converges.

Example: The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ does not converge absolutely because $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series.

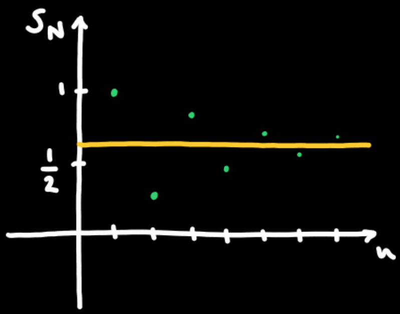
Conditional convergence:

$\boxed{\text{An infinite series } \sum a_n \text{ converges conditionally if } \sum a_n \text{ converges but } \sum |a_n| \text{ diverges.}}$

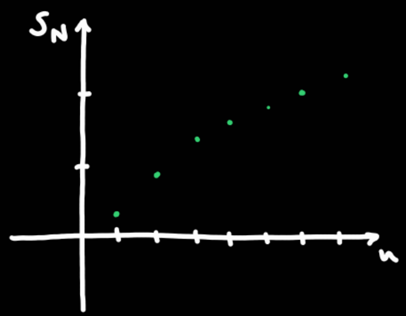
Leibniz test for alternating series:

Assume that $\{a_n\}$ is a positive sequence that is decreasing and converges to zero. Then the alternating series $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges, $0 < S < a_1$, and $S_{2N} < S < S_{2N+1}$ for all positive integer N.

Example: The infinite series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges conditionally. The terms $a_n = \frac{1}{\sqrt{n}}$ are positive and decreasing, and $\lim_{n\to\infty} a_n = 0$. Therefore the series converges to some positive number S by the Leibniz test, and $0 \leq S \leq 1$ since $a_1 = 1$. However the positive series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it is a p-series with $p = \frac{1}{2} < 1$, and thus divergent. Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is convergent but not absolutely convergent, so it is conditionally convergent.



Partial sums of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$.



Partial sums of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Let $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ where $\{a_n\}$ is a positive decreasing sequence that converges to zero.

Then $|S - S_N| < a_{N+1}$.

The error commited when approximating S by S_N is less than the first omitted term a_{N+1}.

Example: The alternating harmonic series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally. The terms $a_n = \frac{1}{n}$ are positive decreasing and $\lim_{n\to\infty} a_n = 0$, so S converges by the Leibniz test. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so S is conditionally convergent but not absolutely convergent.

Now $|S-S_N| < \frac{1}{N+1}$, and if we want to make an error less than 10^{-3} by choosing an appropriate N we consider the inequality: $\frac{1}{N+1} \leq 10^{-3}$. Solving for N we find $N+1 \geq 10^{-3}$ so $N \geq 999$. Finally, we can check that:

$$|S-S_{999}| < \frac{1}{999+1} = \frac{1}{1000} = 10^{-3}, \text{ the desired error bound.}$$

Only for illustrative purposes, we compute with a calculator $S_{999} \approx 0.69365$. We will see in Section 11.7. that $S = \ln(2) \approx 0.69314$. Then:

$$|S-S_{999}| \approx |\ln(2) - 0.69365| \approx 0.0005 = \frac{1}{2000} < \frac{1}{1000} = 10^{-3}.$$

Section 11.5.: The ratio and root tests.

Ratio test:

> Assume that the limit $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.
>
> (i) If $\rho < 1$ then $\sum a_n$ converges absolutely.
>
> (ii) If $\rho > 1$ then $\sum a_n$ diverges.
>
> (iii) If $\rho = 1$ the test is inconclusive (the series may converge or diverge).

Example: Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converge or diverge? Computing the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0 < 1,$$

so the series $\sum^{\infty} \frac{2^n}{n!}$ converges by the ratio test.

Example: Does $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converge or diverge? Computing the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}\right| = \lim_{n\to\infty} \frac{1}{2}\left|\frac{n^2+2n+1}{n^2}\right| = \frac{1}{2} < 1,$$

so the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges by the ratio test.

Example: Does $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{n!}{1000^n}$ converge or diverge? Computing the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{1000^{n+1}} \cdot \frac{1000^n}{n!}\right| = \lim_{n\to\infty} \frac{n+1}{1000} = \infty,$$

so the series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n!}{1000^n}$ diverges by the ratio test.

Example: Does $\sum_{n=1}^{\infty} n^2$ converge or diverge? Computing the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = \lim_{n\to\infty} \left(1+\frac{2}{n}+\frac{1}{n^2}\right) = 1.$$

However, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$, then $\sum_{n=1}^{\infty} n^2$ does <u>not</u> converge.

Example: Does $\sum_{n=1}^{\infty} n^{-2}$ converge or diverge? Computing the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1.$$

However, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a converging p-series, it does converge.

Root test:

Assume that the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.

(i) If $L<1$, then $\sum a_n$ converges absolutely.

(ii) If $L>1$, then $\sum a_n$ diverges.

(iii) If $L=1$, the test is inconclusive (the series may converge or diverge).

<u>Example:</u> Does $\sum_{n=1}^{\infty}\left(\frac{n}{2n+3}\right)^n$ converge or diverge? Computing the root test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{n}{2n+3}=\frac{1}{2}<1,$$

so the series $\sum_{n=1}^{\infty}\left(\frac{n}{2n+3}\right)^n$ converges by the ratio test.

Section 11.6.: Power series.

A $\boxed{\text{power series}}$ with center c is an infinite series:

$$F(x)=\sum_{n=0}^{\infty}a_n\cdot(x-c)^n=a_0+a_1\cdot(x-c)+a_2\cdot(x-c)^2+\cdots$$

where x is a variable. The convergence of a power series depends on the value of x.

<u>Example:</u> Consider $F(x)=1+(x-2)+2\cdot(x-2)^2+3\cdot(x-2)^3+\cdots=1+(x-2)+\sum_{n=2}^{\infty}n\cdot(x-2)^n$. It is a power series with center $c=2$. For $x=\frac{9}{4}$, the power series converges by the ratio test:

$$F\left(\tfrac{9}{4}\right)=1+\left(\tfrac{9}{2}-2\right)+2\cdot\left(\tfrac{9}{2}-2\right)^2+\left(\tfrac{9}{2}-2\right)^3+\cdots=1+\frac{1}{4}+2\cdot\left(\frac{1}{4}\right)^2+3\cdot\left(\frac{1}{4}\right)^3+\cdots$$

For $x=3$, the power series diverges since the general term does <u>not</u> tend to zero:

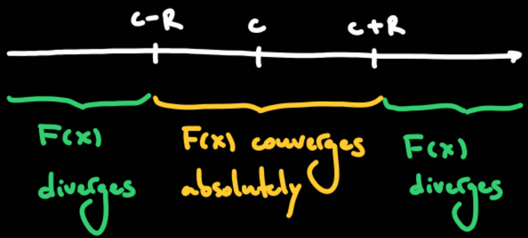
$$F(3)=1+(3-2)+2\cdot(3-2)^2+3\cdot(3-2)^3+\cdots=1+1+2+3+\cdots$$

<u>Radius of convergence:</u>

A power series $F(x)=\sum_{n=0}^{\infty}a_n\cdot(x-c)^n$ has radius of convergence R, which is either a positive

A power series $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence R, which is either a positive number or infinity. If R is finite, $F(x)$ converges absolutely for $|x-c|<R$ and diverges for $|x-c|>R$. If R is infinity, $F(x)$ converges absolutely for all values of x.

The domain where $F(x)$ converges is called $\boxed{\text{interval of convergence}}$ and consists of the open interval $(c-R, c+R)$, sometimes with one or both endpoints $c-R$ and $c+R$.



There are two steps in determining the interval of convergence of a power series $F(x)$:

1. Find the radius of convergence R.

2. Check whether we have convergence or divergence at the endpoints.

<u>Example:</u> Determine the interval of convergence of $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}$.

1. Find the radius of convergence. We compute the ratio ρ of the ratio test, with $a_n = \frac{x^n}{2^n}$:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}}\right|\cdot\left|\frac{2^n}{x^n}\right| = \lim_{n\to\infty}\frac{1}{2}|x| = \frac{|x|}{2}.$$

Then $F(x)$ converges when $\rho<1$, so $\frac{|x|}{2}<1$, so $|x|<2$. Similarly, $F(x)$ diverges when $\rho>1$, so $\frac{|x|}{2}>1$, so $|x|>2$. The radius of convergence of $F(x)$ is $R=2$.

2. Check the endpoints. The ratio test is inconclusive for $x=\pm 2$, so we check those cases

directly:

$$F(2)=\sum_{n=0}^{\infty}\frac{2^n}{2^n}=1+1+1+\cdots, \quad F(-2)=\sum_{n=0}^{\infty}\frac{(-2)^n}{2^n}=1-1+1-1+\cdots$$

Since both series diverge, $F(x)$ converges only for $|x|<2$.

<u>Example:</u> Determine the interval of convergence of $F(x)=\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n \cdot n}\cdot(x-5)^n$.

1. Find the radius of convergence. We compute the ratio ρ using $a_n=\frac{(-1)^n\cdot(x-5)^n}{4^n\cdot n}$:

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{4^{n+1}(n+1)}\cdot\frac{4^n\cdot n}{(x-5)^n}\right|=|x-5|\cdot\lim_{n\to\infty}\left|\frac{n}{4(n+1)}\right|=\frac{|x-5|}{4}.$$

Then $F(x)$ converges when $\rho<1$, so $\frac{|x-5|}{4}<1$, so $|x-5|<4$. Similarly, $F(x)$ diverges when $\rho>1$, so $\frac{|x-5|}{4}>1$, so $|x-5|>4$. The radius of convergence of $F(x)$ is $R=4$. Note that the center of $F(x)$ is $c=5$, so the endpoints are $c-R=5-4=1$ and $c+R=5+4=9$

2. Check the endpoints. The ratio test is inconclusive for $x=1$ and $x=9$, so we check:

$$F(9)=\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n\cdot n}\cdot(9-5)^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$ which converges by the Leibniz test.

$$F(1)=\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n\cdot n}\cdot(1-5)^n=\sum_{n=1}^{5}\frac{1}{n}$$ which is a divergent harmonic series.

Then $F(x)$ converges for x in the half-open interval $(1,9]$.

<u>Example:</u> Determine the interval of convergence of $F(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

Although this power series has only even powers of x, we can still use the ratio test with $a_n=\frac{x^{2n}}{(2n)!}$. We obtain $a_{n+1}=\frac{x^{2(n+1)}}{(2(n+1))!}=\frac{x^{2n+2}}{(2n+2)!}$, and since $(2n+2)!=(2n+2)(2n+1)(2n)!$

then:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = |x|^2\cdot\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0.$$

Thus $\rho = 0 < 1$ for all x, so $F(x)$ converges for all x. The radius of convergence is $R = \infty$.

<u>Geometric series:</u> Recall that for $|r|<1$ we have $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. Hence:

$$\boxed{\sum_{n=0}^{\infty} x^n \text{ is a power series with radius of convergence } R=1, \text{ and } \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x|<1.}$$

<u>Example:</u> To see that $\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$ for $|x|<\frac{1}{2}$ we substitute $2x$ for x in the geometric

series: $\frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^n$, and since the original series is valid for $|x|<1$

the modified series is valid for $|2x|<1$, namely $|x|<\frac{1}{2}$.

<u>Example:</u> Find a power series expansion, with center $c=0$, for $\frac{1}{2+x^2}$, and find the interval of

convergence. We can rewrite:

$$\frac{1}{2+x^2} = \frac{1}{2}\cdot\left(\frac{1}{1+\frac{x^2}{2}}\right) = \frac{1}{2}\left(\frac{1}{1-\left(\frac{-x^2}{2}\right)}\right) = \frac{1}{2}\cdot\left(\frac{1}{1-u}\right)$$

$u = \frac{-x^2}{2}$

and substitute in the geometric series:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \text{ for } |u|<1, \text{ so } \frac{1}{2+x^2} = \frac{1}{2}\left(\frac{1}{1-u}\right) = \frac{1}{2}\cdot\sum_{n=0}^{\infty} u^n = \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{-x^2}{2}\right)^n =$$

$$= \frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^n}{2^{n+1}} \text{ for } \left|\frac{-x^2}{2}\right|<1,$$

namely $|x|<\sqrt{2}$.

Thus $\frac{1}{2+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^n}{2^{n+1}}$ for $|x|<\sqrt{2}$, so the interval of convergence is $(-\sqrt{2},\sqrt{2})$.

Term-by-term differentiation and integration:

Assume that $F(x) = \sum_{n=0}^{\infty} a_n \cdot (x-c)^n$ is a power series with radius of convergence $R > 0$. Then $F(x)$ is differentiable on $(c-R, c+R)$ if $R < \infty$, and for all x if $R = \infty$. We can integrate and differentiate term by term, so for x in $(c-R, c+R)$:

$$F'(x) = \sum_{n=1}^{\infty} n \cdot a_n \cdot (x-c)^{n-1}$$

$$\int F(x)\,dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot (x-c)^{n+1}, \quad A \text{ some contnct.}$$

These series have the same radius of convergence R.

<u>Example:</u> To see that $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$ for $-1 < x < 1$, we will differentiate the geometric series, which has radius of convergence $R = 1$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \text{ so by differentiating term by term for } |x| < 1:$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) = \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \frac{d}{dx}(x^4) + \cdots$$

Which results in:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots, \text{ which converges for } |x| < 1, \text{ namely } -1 < x < 1.$$

<u>Example:</u> To see that $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $-1 < x < 1$, we will integrate the geometric series, which has radius of convergence $R = 1$. Since $\arctan(x)$ is an antiderivative of $\frac{1}{1+x^2}$, we

substitute $-x^2$ in the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{so} \quad \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

where the second expansion is valid for $|-x^2| < 1$, namely $|x| < 1$. Now integrating term by term:

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx =$$

$$= \int dx + \int (-x^2)\,dx + \int x^4\,dx + \int (-x^6)\,dx + \cdots = A + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

which converges for $|x| < 1$. Since $0 = \arctan(0) = A$, we have:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } -1 < x < 1.$$

Section 11.7.: Taylor series

Taylor series expansion:

> If $f(x)$ is represented by a power series centered at c in an interval $|x-c| < R$ for some $R > 0$,
>
> then that power series is the Taylor series: $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

When $c = 0$, we call $T(x)$ the **Maclaurin series**.

Example: Find the Taylor series for $f(x) = x^{-3}$ centered at $c = 1$.

The derivatives of $f(x)$ are:

$$f'(x) = -3x^{-4}, \quad f''(x) = (-3)(-4)x^{-5}, \ldots, \quad f^{(n)}(x) = (-1)^n \cdot 3 \cdot 4 \cdots (n+2) \cdot x^{-3-n}.$$

Since $3 \cdot 4 \cdots (n+2) = \frac{(n+2)!}{2}$, then $f^{(n)}(1) = (-1)^n \cdot \frac{(n+2)!}{2}$. Since $(n+2)! = (n+2)(n+1) \cdot n!$, we have $a_n = \dfrac{f^{(n)}(1)}{n!} = \dfrac{(-1)^n \cdot \frac{(n+2)!}{2}}{n!} = (-1)^n \cdot \dfrac{(n+2)(n+1)\,n!}{2 \cdot n!} = (-1)^n \cdot \dfrac{(n+2)(n+1)}{2}$. Hence the

Taylor series of $f(x) = x^{-3}$ centered at $c=1$ is:

$$T(x) = 1 - 3(x-1) + 6(x-1)^2 - 10\cdot(x-1)^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(n+2)(n-1)}{2} \cdot (x-1)^n .$$

Given a function $f(x)$, there is <u>no</u> guarantee that $T(x)$ converges to $f(x)$, not even if $T(x)$ converges.

Let $I = (c-R, c+R)$ with $R>0$, and suppose that there exists a $k>0$ such that:

$$|f^{(n)}(x)| \leq k \quad \text{for all } n \geq 0 \text{ and all } x \in I \quad (\text{all the derivatives of } f(x) \text{ are bounded by } k).$$

Then $f(x)$ is represented by its Taylor series in I:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} \cdot (x-c)^n \quad \text{for all } x \text{ in } I.$$

<u>Example:</u> The Maclaurin expansion $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ is valid for

all x. Recall that for $f(x) = \sin(x)$ we have a periodic derivative:

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f'''(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x).$$

Evaluating at $x=0$ we have:

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f^{(4)}(0) = 0.$$

The even derivatives are always zero, and the odd derivatives alternate sign. Hence the

non-zero Taylor coefficients are $a_{2n+1} = \dfrac{(-1)^n}{(2n+1)!}$, and $|f^{(n)}(x)| \leq 1$ for all $n \geq 0$ and all

real number x, so using $k=1$ in the above result we have:

$$f(x)=\sum_{n=0}^{\infty} a_{2n+1}\cdot x^{2n+1}=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \text{ for all } x.$$

Example: The Maclaurin expansion $\cos(x)=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$ is valid for all x.

Recall that for $f(x)=\cos(x)$ we have a periodic derivative:

$$f'(x)=-\sin(x),\quad f''(x)=-\cos(x),\quad f'''(x)=\sin(x),\quad f^{(4)}(x)=\cos(x).$$

Evaluating at $x=0$ we have:

$$f(0)=1,\quad f'(0)=0,\quad f''(0)=-1,\quad f'''(0)=0,\quad f^{(4)}(0)=1.$$

The odd derivatives are always zero, and the even derivatives alternate sign. Hence the non-zero Taylor coefficients are $a_{2n}=\frac{(-1)^n}{(2n)!}$, and $|f^{(n)}(x)|\le 1$ for all $n\ge 0$ and all real number x, so using $k=1$ in the above result we have:

$$f(x)=\sum_{n=0}^{\infty} a_{2n}\cdot x^{2n}=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \text{ for all } x.$$

Example: The Taylor series of $f(x)=e^x$ at $x=c$ is $T(x)=\sum_{n=0}^{\infty}\frac{e^c}{n!}\cdot(x-c)^n$, and is valid for all x.

Recall that $f^{(n)}(c)=e^c$ for all $n\ge 0$, so indeed $a_n=\frac{e^c}{n!}$. Since $f^{(n)}(x)=e^x$ is increasing, for any $R>0$ we have $|f^{(n)}(x)|\le e^{c+R}$ for all x in $(c-R,c+R)$. Hence using $k=e^{c+R}$ in the above result, we obtain that $T(x)$ converges to $f(x)$ for all x in $(c-R,c+R)$. Since R is arbitrary, the Taylor expansion holds for all x.

Example: Find the Maclaurin series for $f(x)=x^2e^x$

Example: Find the Maclaurin series for $f(x) = x^2 \cdot e^x$.

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ then:

$$x^2 \cdot e^x = x^2 \cdot \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} .$$

Example: Find the Maclaurin series for e^{-x^2}.

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ then by substituting $-x^2$:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots .$$

Example: Find the Maclaurin series for $f(x) = \ln(1+x)$.

Since $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$, by integrating term by term we obtain:

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx = \int dx - \int x\,dx + \int x^2\,dx - \int x^3\,dx + \cdots =$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n} \quad \text{for } |x| < 1,$$

where the constant of integration is zero because $\ln(1+0) = \ln(1) = 0$. This Maclaurin series also holds for $x = 1$.

Example: Compute the fifth Maclaurin polynomial of $f(x) = e^x \cdot \cos(x)$.

We multiply the fifth Maclaurin polynomials of e^x and $\cos(x)$, and we ignore the terms of degree greater than five:

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \text{ gives:}$$

$$\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}\right)-\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)\left(\frac{x^2}{2}\right)+(1+x)\left(\frac{x^4}{24}\right)=1+x-\frac{x^3}{3}-\frac{x^4}{6}-\frac{x^5}{30}.$$

Then $T_5(x)=1+x-\frac{x^3}{3}-\frac{x^4}{6}-\frac{x^5}{30}$ is the fifth Maclaurin polynomial for $f(x)=e^x\cdot\cos(x)$.

<u>The binomial series:</u> For any exponent a and for $|x|<1$:

$$\boxed{(1+x)^a=1+\frac{a}{1!}x+\frac{a\cdot(a-1)}{2!}x^2+\frac{a\cdot(a-1)\cdot(a-2)}{3!}x^3+\cdots=\sum_{n=0}^{\infty}\binom{a}{n}x^n}$$

where $\binom{a}{n}=\frac{a\cdot(a-1)(a-2)\cdots(a-n+1)}{n!}$.

<u>Example:</u> Find the Maclaurin series for $f(x)=\frac{1}{\sqrt{1-x^2}}$.

We first compute the Maclaurin series for $\frac{1}{\sqrt{1+x}}$, we will then substitute $-x^2$. The coefficients in the binomial series for $(1+x)^{\frac{-1}{2}}$ are:

$$1,\quad \frac{-\frac{1}{2}}{1!}=\frac{-1}{2},\quad \frac{\left(\frac{-1}{2}\right)\cdot\left(\frac{-3}{2}\right)}{2!}=\frac{1\cdot 3}{2\cdot 4},\quad \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}=-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6},\ \ldots$$

where the general term is:

$$\binom{\frac{-1}{2}}{n}=\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\cdots\left(\frac{-(2n-1)}{2}\right)}{n!}=(-1)^n\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}.$$

Hence for $|x|<1$:

$$\frac{1}{\sqrt{1+x}}=1+\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot x^n=1-\frac{x}{2}+\frac{3}{8}x^2-\cdots$$

and if $|x|<1$ then $|-x^2|=|x|^2<1$ so substituting $-x^2$ for x we obtain:

$$\frac{1}{\sqrt{1-x^2}}=1+\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot(-x^2)^n=1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\cdot x^{2n}=1+\frac{x^2}{2}+\frac{3}{8}x^4+\cdots$$