Math 33A
Linear Algebra and Applications

Discussion for June 27-July 1, 2022

Problem 1.

We say that two $n \times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. Give an example of two 2×3 matrices of the same type. Give an example of two 2×3 matrices of different type.

Solution: Two matrices of the same type are

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & \pi \\ 0 & 1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}.$$

Two matrices of different type are

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 2($\star$).

How many types of 2×2 matrices in reduced row-echelon form are there?

Solution: Four

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Problem 3.

How many types of 3×2 matrices in reduced row-echelon form are there?

Solution: Four

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & k \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & k \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Problem 4.

Suppose you apply Gauss–Jordan elimination to a matrix. Explain how you can be sure that the resulting matrix is in reduced row-echelon form.

Solution: Each one of the three row operations in the process of Gauss-Jordan elimination corresponds to one of the three conditions required for a matrix to be in reduced row-echelon form. Being able to divide by a number allows us to have leading ones, subtracting rows allows us to put zeros under leading ones, and swapping rows allows us to rearrange the matrix. We can then bring to the bottom the rows that may not have leading ones, and when fixing a leading one, we can put all the leading ones lying to the left of this fixed leading one above it.

Problem 5.

Suppose matrix A is transformed into matrix B by means of an elementary row operation. Is there an elementary row operation that transforms B into A? Explain.

Solution: Yes, elementary row operations are reversible, so we can undo them.

Problem 6.

Suppose matrix A is transformed into matrix B by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms B into A? Explain.

Solution: Yes, we should do the inverse operations, in the reverse order (so if we swapped, then divided, then added two rows, we should first add the rows, then multiply, then swap).

Problem 7.

Consider an $n \times m$ matrix A. Can you transform $\mathrm{rref}(A)$ into A by a sequence of elementary row operations? Explain.

Solution: Yes, $\mathrm{rref}(A)$ is obtained by elementary row operations, so by undoing them we can reverse the process.

Problem 8.

Show that if T is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

$$T\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 T(\vec{e_1}) + \cdots + x_m T(\vec{e_m}),$$

where $\vec{e_1}, \ldots, \vec{e_m}$ are the standard vectors in $\mathbb{R}^m$.

Solution: We can rewrite

$$T\begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix} = T\left(\begin{bmatrix}x_1\\0\\ \vdots\\0\end{bmatrix}+\begin{bmatrix}0\\x_2\\ \vdots\\0\end{bmatrix}+\cdots+\begin{bmatrix}0\\0\\ \vdots\\x_m\end{bmatrix}\right)=T\left(x_1\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\ \vdots\\0\end{bmatrix}+\cdots+x_m\begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}\right)=$$
$$=T(x_1\vec{e_1}+\cdots+x_m\vec{e_m})=T(x_1\vec{e_1})+\cdots+T(x_m\vec{e_m})=x_1T(\vec{e_1})+\cdots+x_mT(\vec{e_m}),$$

where the last two equalities are using that T is linear.

Problem 9($\star$).
Describe all linear transformations from $\mathbb{R}$ to $\mathbb{R}$. What do their graphs look like?

Solution: The linear transformations are of the form $[y] = [a][x]$ for some real number a. They encode the equation $y = ax$, which are lines through the origin.

Problem 10.
Describe all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$. What do their graphs look like?

Solution: The linear transformations are of the form $[z] = \begin{bmatrix}a & b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ for some real numbers a, b. They encode the equation $z = ax + by$, which is a plane through the origin.

Problem 11.
Consider two linear transformations $\vec{y} = T(\vec{x})$ and $\vec{z} = L(\vec{y})$, where T goes from $\mathbb{R}^m$ to $\mathbb{R}^p$ and L goes from $\mathbb{R}^p$ to $\mathbb{R}^n$. Is the transformation $\vec{z} = L(T(\vec{x}))$ linear as well?

Solution: Yes. We can check that for vectors $\vec{x}, \vec{x_1}, \vec{x_2}$ and constant k we have

$$\begin{aligned}L(T(\vec{x_1}+\vec{x_2})) =& L(T(\vec{x_1})+T(\vec{x_2})) = L(T(\vec{x_1})+L(T(\vec{x_2}))\\ L(T(k\vec{x})) =& L(kT(\vec{x})) = kL(T(\vec{x}))\end{aligned}$$

so the transformation LT is linear.

Problem 12.
Let

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}p & q\\ r & s\end{bmatrix}.$$

Find the matrix of the linear transformation $T(\vec{x}) = B(A\vec{x})$.

Solution: We apply T to the vectors $\vec{e_1}$ and $\vec{e_2}$, and then T will have matrix $[T(\vec{e_1})\, T(\vec{e_2})]$. Since

$$T\begin{bmatrix}1\\0\end{bmatrix} = B\left(A\begin{bmatrix}1\\0\end{bmatrix}\right) = B\left(\begin{bmatrix}a\\c\end{bmatrix}\right) = \begin{bmatrix}pa+qc\\ra+sc\end{bmatrix}$$
$$T\begin{bmatrix}0\\1\end{bmatrix} = B\left(A\begin{bmatrix}0\\1\end{bmatrix}\right) = B\left(\begin{bmatrix}b\\d\end{bmatrix}\right) = \begin{bmatrix}pb+qd\\rb+sd\end{bmatrix}$$

the associated matrix is

$$\begin{bmatrix}pa+qc & pb+qd\\ra+sc & rb+sd\end{bmatrix}.$$