Math 33A
Linear Algebra and Applications

Discussion for July 4-8, 2022

Problem 1.

Show that if a square matrix A has two equal columns, then A is not invertible.

Problem 2($\star$).

Which of the following linear transformations T from $\mathbb{R}^3$ to $\mathbb{R}^3$ are invertible? Find the inverse if it exists.

(a) Reflection about a plane.

(b) Orthogonal projection onto a plane.

(c) Scaling by a real factor (namely, fix a real number r and consider $T(\vec{v}) = r\vec{v}$, for all vectors $\vec{v}$).

(d) Rotation about an axis.

Problem 3.

A square matrix is called a permutation matrix if it contains a 1 exactly once in each row and in each column, with all other entries being 0. Give an example of two different 3×3 permutation matrices.

Problem 4.

Are permutation matrices invertible? If so, is the inverse a permutation matrix as well?

Problem 5.

Consider two invertible $n \times n$ matrices A and B. Is the linear transformation $\vec{y} = A(B(\vec{x}))$ invertible? If so, what is the inverse?

Problem 6.

Are the columns of an invertible matrix linearly independent?

Problem 7.

Consider linearly independent vectors $\vec{v_1}, \ldots, \vec{v_m}$ in $\mathbb{R}^n$, and let A be an invertible $m \times m$ matrix. Are the columns of the following matrix linearly independent?

$$\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix} A$$

Problem 8.

Consider a matrix A of the form

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

where $a^2 + b^2 = 1$ and $a \neq 1$. Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis

$$\begin{bmatrix} b \\ 1-a \end{bmatrix}, \begin{bmatrix} a-1 \\ b \end{bmatrix}.$$

Interpret the answer geometrically.

Problem 9.

Let A and B be square matrices, if there is an invertible matrix S such that $B = S^{-1}AS$ we say that A is similar to B. Find an invertible 2×2 matrix S such that

$$S^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} S$$

is of the form

$$\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}.$$

What can you say about two of those matrices?

Problem 10.

If A is a 2×2 matrix such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix},$$

show that A is similar to a diagonal matrix D. Find an invertible S such that $S^{-1}AS = D$.

Problem 11.

If $c \neq 0$, find the matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

with respect to the basis

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix}.$$

Problem 12($\star$).

Is there a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that $\mathfrak{B}$-matrix B of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

is upper triangular?