Math 33A
Linear Algebra and Applications

Discussion for July 4-8, 2022

Problem 1.

Show that if a square matrix A has two equal columns, then A is not invertible.

Solution: Applying an elementary row operation to a matrix with two equal columns will result in a matrix with two equal columns. Say A is an $n \times n$ matrix, then $\text{rref}(A)$ has two equal columns, so it is not I_n. To find the inverse of A, we would compute $\text{rref}([A|I_n])$, but since $\text{rref}(A) \neq I_n$ then $\text{rref}([A|I_n]) \neq [I_n|B]$, so A is not invertible.

Problem 2($\star$).

Which of the following linear transformations T from $\mathbb{R}^3$ to $\mathbb{R}^3$ are invertible? Find the inverse if it exists.

(a) Reflection about a plane.

(b) Orthogonal projection onto a plane.

(c) Scaling by a real factor (namely, fix a real number r and consider $T(\vec{v}) = r\vec{v}$, for all vectors $\vec{v}$).

(d) Rotation about an axis.

Solution:

1. Invertible, this transformation is its own inverse.
2. Not invertible, if $\vec{v}$ is a vector perpendicular to the plane, then all the vectors $k\vec{v}$ for k a real number are sent to the same vector.
3. Invertible, the inverse is scaling by $1/r$.
4. Invertible, the inverse is rotating about the same axis in the opposite direction.

Problem 3.

A square matrix is called a permutation matrix if it contains a 1 exactly once in each row and in each column, with all other entries being 0. Give an example of two different 3×3 permutation matrices.

Solution: Two different permutation matrices are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Problem 4.

Are permutation matrices invertible? If so, is the inverse a permutation matrix as well?

> **Solution:** Yes. Yes. Let A be an $n \times n$ permutation matrix, then to go from $\text{rref}([A|I_n])$ to $\text{rref}([I_n|B])$ we only need to permute rows, so B will have a 1 exactly once in each row and column.

Problem 5.

Consider two invertible $n \times n$ matrices A and B. Is the linear transformation $\vec{y} = A(B(\vec{x}))$ invertible? If so, what is the inverse?

> **Solution:** Yes, the inverse is $\vec{x} = B^{-1}(A^{-1}(\vec{y}))$.

Problem 6.

Are the columns of an invertible matrix linearly independent?

> **Solution:** Yes. If A is invertible then $\ker(A) = \{0\}$ so its columns are linearly independent.

Problem 7.

Consider linearly independent vectors $\vec{v_1}, \ldots, \vec{v_m}$ in $\mathbb{R}^n$, and let A be an invertible $m \times m$ matrix. Are the columns of the following matrix linearly independent?

$$\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix} A$$

> **Solution:** Yes. If A is invertible then $\ker(A) = \{\vec{0}\}$ so the columns of A are linearly independent. Also, since all the vectors $\vec{v_1}, \ldots, \vec{v_m}$ are linearly independent, then $\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix}$ has kernel $\{\vec{0}\}$. If $\vec{x}$ is in the kernel of $\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix} A$ then $A\vec{x}$ is in the kernel of $\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix}$, so $A\vec{x} = \vec{0}$. Thus $\vec{x}$ is in the kernel of A, so $\vec{x} = \vec{0}$.

This means that the matrix $\begin{bmatrix} | & & | \\ \vec{v_1} & \cdots & \vec{v_m} \\ | & & | \end{bmatrix} A$ has kernel $\{\vec{0}\}$, so all its columns are linearly independent.

Problem 8.

Consider a matrix A of the form

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

where $a^2 + b^2 = 1$ and $a \neq 1$. Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis

$$\begin{bmatrix} b \\ 1-a \end{bmatrix}, \begin{bmatrix} a-1 \\ b \end{bmatrix}.$$

Interpret the answer geometrically.

Solution: There are two ways of seeing this, one more geometric, the other more algebraic. Geometrically, the vector $\vec{v_1} = \begin{bmatrix} b \\ 1-a \end{bmatrix}$ determines a line in $\mathbb{R}^2$, and the vector $\vec{v_2} = \begin{bmatrix} a-1 \\ b \end{bmatrix}$ is perpendicular to this line. The matrix A is representing a reflection about the line parallel to $\vec{v_1}$. In the basis $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}\}$ a reflection about this line keeps $\vec{v_1}$ untouched and changes the sign of $\vec{v_2}$, and thus a reflection about this line has matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Algebraically, the matrix is given by applying the linear transformation to $\vec{v_1}$ and putting the result in the first column, and then applying the linear transformation to $\vec{v_2}$ and putting the result in the second column, giving

$$\begin{aligned} \begin{bmatrix} [T(\vec{v_1})]_{\mathfrak{B}} & [T(\vec{v_2})]_{\mathfrak{B}} \end{bmatrix} &= \begin{bmatrix} \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} b \\ 1-a \end{bmatrix}\right]_{\mathfrak{B}} & \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1-a \\ -b \end{bmatrix}\right]_{\mathfrak{B}} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} ab+b-ba \\ b^2+a^2-a \end{bmatrix}_{\mathfrak{B}} & \begin{bmatrix} a^2+b^2-a \\ ba-b-ab \end{bmatrix}_{\mathfrak{B}} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} b \\ 1-a \end{bmatrix}_{\mathfrak{B}} & \begin{bmatrix} 1-a \\ -b \end{bmatrix}_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Problem 9.

Let A and B be square matrices, if there is an invertible matrix S such that $B = S^{-1}AS$

we say that A is similar to B. Find an invertible 2×2 matrix S such that

$$S^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} S$$

is of the form

$$\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}.$$

What can you say about two of those matrices?

Solution: Since S is a 2×2 matrix, it has four unknowns. Leaving b and d representing any two real numbers, we have the equation

$$\frac{1}{xw - yz} \begin{bmatrix} w & -y \\ -z & x \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$$

which (interestingly enough, see Problem 11 for more details about this) forces $b = 2$ and $d = 5$. Setting x and w as free variables, these four equations impose the restrictions $2y = w - x$ and $4z = w - 3x$. Since S has to be invertible, we have the additional restriction $xw - yz = \det(S) \neq 0$, which with the above solutions becomes $w^2 - 12wx + 3x^2 \neq 0$. Thus, as long as this invertibility condition is satisfied, we have

$$S = \begin{bmatrix} x & \frac{w-x}{2} \\ \frac{w-3x}{4} & w \end{bmatrix}.$$

The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is similar to the matrix $\begin{bmatrix} 0 & 2 \\ 1 & 5 \end{bmatrix}$.

Problem 10.

If A is a 2×2 matrix such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix},$$

show that A is similar to a diagonal matrix D. Find an invertible S such that $S^{-1}AS = D$.

Solution: Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by A, namely $T(\vec{x}) = A\vec{x}$. Since we are given the image of $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, consider the basis $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}\}$. The matrix of T with respect to $\mathfrak{B}$ is

$$D = \begin{bmatrix} [T(\vec{v_1})]_{\mathfrak{B}} & [T(\vec{v_2})]_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}_{\mathfrak{B}} & \begin{bmatrix} -2 \\ -1 \end{bmatrix}_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} S^{-1} \begin{bmatrix} 3 \\ 6 \end{bmatrix} & S^{-1} \begin{bmatrix} -2 \\ -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since we have changed from the standard basis to a new basis, we have $A = SDS^{-1}$, and thus $D = S^{-1}AS$ so A is similar to D.

Problem 11.

If $c \neq 0$, find the matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

with respect to the basis

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix}.$$

Solution: Denote this basis $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}\}$, the matrix of T with respect to $\mathfrak{B}$ is

$$\begin{aligned}
\big[[T(\vec{v_1})]_{\mathfrak{B}} \quad [T(\vec{v_2})]_{\mathfrak{B}}\big] &= \left[\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_{\mathfrak{B}} \quad \left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix}\right]_{\mathfrak{B}}\right] \\
&= \left[\begin{bmatrix} a \\ c \end{bmatrix}_{\mathfrak{B}} \quad \begin{bmatrix} a^2+bc \\ ac+cd \end{bmatrix}_{\mathfrak{B}}\right] = \left[\begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix}^{-1}\begin{bmatrix} a \\ c \end{bmatrix} \quad \begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix}^{-1}\begin{bmatrix} a^2+bc \\ ac+cd \end{bmatrix}\right] \\
&= \left[\begin{bmatrix} 1 & -a/c \\ 0 & 1/c \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} \quad \begin{bmatrix} 1 & -a/c \\ 0 & 1/c \end{bmatrix}\begin{bmatrix} a^2+bc \\ ac+cd \end{bmatrix}\right] = \begin{bmatrix} 0 & bc-ad \\ 1 & a+d \end{bmatrix}.
\end{aligned}$$

This explains what is going on in Problem 2. Setting $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, via a change of basis it will be similar to a matrix of the form $\begin{bmatrix} 0 & bc-ad \\ 1 & a+d \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 5 \end{bmatrix}$, forcing the mysterious appearance of the column $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Moreover, this forces $\vec{v_2} = \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. In particular, using as basis the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, we have that $S = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ is a solution for Problem 9.

Problem 12($\star$).

Is there a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that $\mathfrak{B}$-matrix B of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$$

is upper triangular?

Solution: No. Note first that T is a rotation of angle $\pi/2$. Note second that if T could be written as an upper triangular matrix in the basis $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}\}$ that would mean that $T(\vec{v_1}) = k\vec{v_1}$ for some real scalar k. In other words, $T(\vec{v_1})$ would be parallel to $\vec{v_1}$. However, since T is a rotation, this is impossible.