

**Math 33A**  
**Linear Algebra and Applications**

**Discussion for July 11-14, 2022**

**Problem 1.**

Here is an infinite-dimensional version of Euclidean space: In the space of all infinite sequences, consider the subspace  $\ell_2$  of square-summable sequences (namely, those sequences  $(x_1, x_2, \dots)$  for which the infinite series  $x_1^2 + x_2^2 + \dots$  converges). For  $x$  and  $y$  in  $\ell_2$ , we define

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots} \quad \text{and} \quad \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \dots$$

A preliminary question is, why do  $\|\vec{x}\|$  and  $\vec{x} \cdot \vec{y}$  make sense, that is, why are they finite real numbers?

- Check that  $\vec{x} = (1, 1/2, 1/4, 1/8, 1/16, \dots)$  is in  $\ell_2$ , and find  $\|\vec{x}\|$ . Recall the formula for the geometric series:  $1 + a + a^2 + a^3 + \dots = 1/(1 - a)$  if  $-1 < a < 1$ .
- Find the angle between  $(1, 0, 0, 0, \dots)$  and  $(1, 1/2, 1/4, 1/8, \dots)$ .
- Give an example of a sequence  $(x_1, x_2, \dots)$  that converges to 0 ( $\lim_{n \rightarrow \infty} x_n = 0$ ) but does not belong to  $\ell_2$ .
- Let  $L$  be the subspace of  $\ell_2$  spanned by  $(1, 1/2, 1/4, 1/8, \dots)$ . Find the orthogonal projection of  $(1, 0, 0, 0, \dots)$  onto  $L$ .

The Hilbert space  $\ell_2$  was initially used mostly in physics: Werner Heisenberg's formulation of quantum mechanics is in terms of  $\ell_2$ . Today, this space is used in many other applications, including economics. See, for example, the work of the economist Andreu Mas-Colell of the University of Barcelona.

**Problem 2.**

Give an algebraic proof for the triangle inequality

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$$

Draw a sketch.

**Problem 3.**

- Consider a vector  $\vec{v}$  in  $\mathbb{R}^n$ , and a scalar  $k$ . Show that  $\|k\vec{v}\| = |k|\|\vec{v}\|$ .
- Show that if  $\vec{v}$  is a nonzero vector in  $\mathbb{R}^n$ , then  $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$  is a unit vector.

**Problem 4(★).**

Can you find a line  $L$  in  $\mathbb{R}^n$  and a vector  $\vec{x}$  in  $\mathbb{R}^n$  such that  $\vec{x} \cdot \text{proj}_L \vec{x}$  is negative? Explain, arguing algebraically.

**Problem 5(★).**

The following is one way to define the quaternions, discovered in 1843 by the Irish mathematician Sir W. R. Hamilton. Consider the set  $H$  of all  $4 \times 4$  matrices  $M$  of the form

$$M = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$$

where  $p, q, r, s$  are arbitrary real numbers. We can write  $M$  more succinctly in partitioned form as

$$M = \begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix}$$

where  $A$  and  $B$  are rotation-scaling matrices.

- (a) Show that  $H$  is closed under addition: If  $M$  and  $N$  are in  $H$ , then so is  $M + N$ .
- (b) Show that  $H$  is closed under scalar multiplication: If  $M$  is in  $H$  and  $k$  is an arbitrary scalar, then  $kM$  is in  $H$ .
- (c) The above show that  $H$  is a subspace of the linear space  $\mathbb{R}^{4 \times 4}$ . Find a basis of  $H$ , and thus determine the dimension of  $H$ .
- (d) Show that  $H$  is closed under multiplication: If  $M$  and  $N$  are in  $H$ , then so is  $MN$ .
- (e) Show that if  $M$  is in  $H$ , then so is  $M^T$ .
- (f) For a matrix  $M$  in  $H$ , compute  $M^T M$ .
- (g) Which matrices  $M$  in  $H$  are invertible? If a matrix  $M$  in  $H$  is invertible, is  $M^{-1}$  necessarily in  $H$  as well?
- (h) If  $M$  and  $N$  are in  $H$ , does the equation  $MN = NM$  always hold?

**Problem 6.**

Consider a consistent system  $A\vec{x} = \vec{b}$ .

- (a) Show that this system has a solution  $\vec{x}_0$  in  $(\ker A)^\perp$ . Justify why an arbitrary solution  $\vec{x}$  of the system can be written as  $\vec{x} = \vec{x}_h + \vec{x}_0$ , where  $\vec{x}_h$  is in  $\ker(A)$  and  $\vec{x}_0$  is in  $(\ker A)^\perp$ .
- (b) Show that the system  $A\vec{x} = \vec{b}$  has only one solution in  $(\ker A)^\perp$ .
- (c) If  $\vec{x}_0$  is the solution in  $(\ker A)^\perp$  and  $\vec{x}_1$  is another solution of the system  $A\vec{x} = \vec{b}$ , show that  $\|\vec{x}_0\| < \|\vec{x}_1\|$ . The vector  $\vec{x}_0$  is called the minimal solution of the linear system  $A\vec{x} = \vec{b}$ .

**Problem 7.**

Define the term minimal least-squares solution of a linear system. Explain why the minimal least-squares solution  $\vec{x}^*$  of a linear system  $A\vec{x} = \vec{b}$  is in  $(\ker A)^\perp$ .